Long-range memory in nonlinear GARCH(1,1) models

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Introduction

Econophysics started out as purely empirical science. Physicists around the world analyzed huge amounts of financial market data and discovered certain statistical regularities, which is now often referred to as stylized facts [1, 2]. One of these regularities is long-range, decaying as power-law, correlations often referred to as long-range memory [2]. PSD of the time series with long-range memory behaves as $1/f^\beta$ with β values being close to 1. At the same time economists proposed, applied and further developed auto-regressive conditionally heteroskedastic (ARCH) models [3].

Abbreviations: GARCH - generalized ARCH; SDE - stochastic differential equation; PDF - probability density function; PSD - power spectral density.

SDE reproducing long-range memory

Previously in [4] a SDE,

$$\mathrm{d}\,x = \sigma^2 \left(\eta - \frac{1}{2}\lambda\right) x^{2\eta-1}\,\mathrm{d}\,t + \sigma x^\eta \,\mathrm{d}\,W_t, \tag{1}$$

was proposed to model $1/f$ noise, which is often related to a concept of long-range memory. This happens because Wiener-Khinchin theorem implies that in such case auto-correlation function of the signal does not decay or decays as a power-law function with exponent close to 0. This SDE reproduces not only $1/f$ noise, but general power-law PSD with variable exponent β:

$$S(f) \sim \frac{1}{f^\beta}, \quad \beta = 1 + \frac{\lambda - 3}{2\eta - 2}, \quad 0.5 < \beta < 2. \tag{2}$$

Note that Eq. (1) does not satisfy Lipschitz conditions for large x and stationary PDF of Eq. (1) diverges for small x, thus we implement reflective boundary conditions and place them at x_{min} and x_{max}. This results in Eq. (2) holding for the certain range of frequencies - $f_{min} < f < f_{max}$). This is rather natural property as pure $1/f$ noise would imply that signal has infinite power.

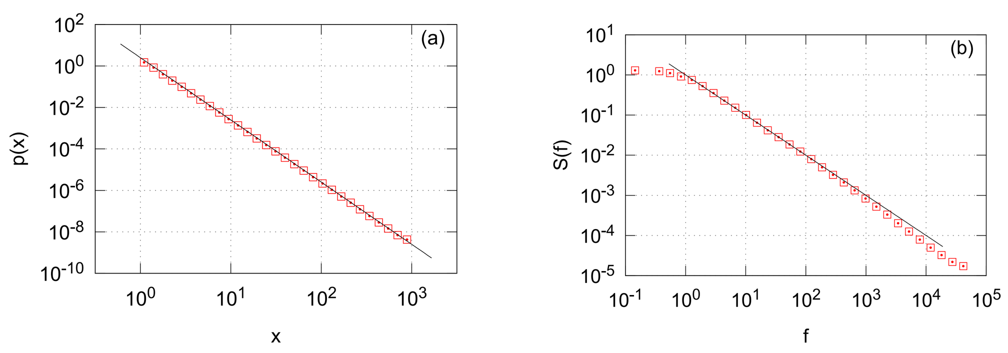

Figure 1: Statistical properties, PDF (a) and PSD (b), obtained by numerically solving SDE (1) (red squares). Black curves show power-law approximations: (a) x^{-3} and (b) $1/f$. Used parameter set: $\eta = 2$, $\lambda = 3$, $x_{\min} = 1$, $x_{\max} = 10^3$, $\sigma = 1$. QR code will take you to interactive HTML5+Javascript app on Physics of Risk (http://mokslasplius.lt/rizikos-fizika) website.

Linear GARCH(1,1) model

The basic idea behind ARCH family models lies in an assumption that certain heteroskedastic economical observables, z_t, may be modeled as being composed of two parts - stochastic part, ω_t, and its time dependent volatility, σ_t:

$$z_t = \sigma_t \omega_t. \tag{3}$$

Stochastic part may be usually assumed to be a simple noise (values of which follow Gaussian distribution, though it may depend on actual application). Time dependent part is assumed to be driven by iterative process.

GARCH(1,1) model is based on the following iterative process:

$$\sigma_t^2 = a + b\sigma_{t-1}^2 \omega_{t-1}^2 + c\sigma_{t-1}^2. \tag{4}$$

In this case new σ_t values depend only on the current state, σ_{t-1}. As σ_t does not depend on past history of system evolution, the model intuitively appears to be memory-less. It exhibits Markov property.

In the diffusion limit, iterative process behind GARCH(1,1) process, Eq. (4), may be rewritten as SDE:

$$\mathrm{d}\,y = (A - Cy)\,\mathrm{d}\,t + |B| y \,\mathrm{d}\,W_t, \tag{5}$$

where W_t is Wiener process (standard one dimensional Brownian motion). Parameters of this SDE are related to parameters of Eq. (4) as follows: $Ah = a$, $B^2 h = 2b^2$ and $Ch = 1 - b - c$ (here h is the infinitesimally small time step).

Eq. (5) appears to be a special case of Eq. (1) with $\eta = 1$ and exponential diffusion restriction. While exponential diffusion restriction has a similar as effect as reflective boundary conditions, being limited to $\eta = 1$ is severely restricting - Eq. (2) diverges, but Eq. (5) has the form of Geometric Brownian motion, which is known to produce Brownian-like PSD, $S(f) \sim 1/f^2$. Though we are not able to control PSD, we can control PDF:

$$p(y) \sim y^{-\lambda}, \quad \lambda = 2 + \frac{2C}{B^2} = 2 + \frac{1 - b - c}{b^2}. \tag{6}$$

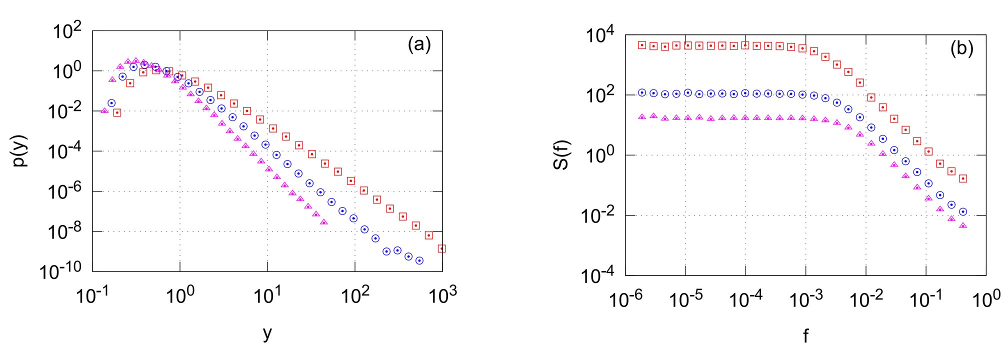

Figure 2: Statistical properties, PDF (a) and PSD (b), of numerically evaluated linear GARCH(1,1) process Eq. (4). Used parameter set: $a = 0.015$, $b = 0.1$, $c = 0.89$ (red squares), 0.88 (blue circles) and 0.87 (magenta triangles). QR code will take you to interactive HTML5+Javascript app on Physics of Risk (http://mokslasplius.lt/rizikos-fizika) website.

Nonlinear modifications of GARCH(1,1)

In order to obtain different η, we propose two nonlinear modifications of Eq. (4):

$$\sigma_t^2 = a + b_1 \sigma_{t-1}^\mu \omega_{t-1}^\mu + c_1 \sigma_{t-1}^2, \tag{7}$$

where $\mu > 2$ is an odd integer, and

$$\sigma_t^2 = a + b_1 \sigma_{t-1}^\mu |\omega_{t-1}|^\mu + \sigma_{t-1}^2 - c_1 \sigma_{t-1}^\mu, \tag{8}$$

where μ may be any positive real number.

In both cases we obtain SDEs which are special cases of Eq. (1) with $2\eta = \mu$ and $\lambda = \mu$. In case of Eq. (7), we have:

$$\mathrm{d}\,y = \left(\frac{A}{y^{\mu-1}} - \frac{C}{y^{\mu-2}}\right) y^{\mu-1}\,\mathrm{d}\,t + |B| y^{\frac{\mu}{2}} \,\mathrm{d}\,W_t, \tag{9}$$

where the parameters are related as follows: $Ah = a$, $Ch = 1 - c$, $B^2 h = \langle \omega^{2\mu} \rangle b^2$. In case of Eq. (8), we have:

$$\mathrm{d}\,y = \left(\frac{A}{y^{\mu-1}} - \frac{C'}{y^{\frac{\mu}{2}-1}}\right) y^{\mu-1}\,\mathrm{d}\,t + |B'| y^{\frac{\mu}{2}} \,\mathrm{d}\,W_t, \tag{10}$$

where the parameters are related as follows: $Ah = a$, $C'h = b\langle|\omega|^\mu\rangle - c$, $B'^2 h = \langle[|\omega|^\mu - \langle|\omega|^\mu\rangle]^2\rangle b^2$. Both of these nonlinear GARCH(1,1) processes are able to reproduce power-law distributions, with exponent $\lambda = \mu$, as well as power-law PSD [5]:

$$S(f) \sim 1/f^\beta, \quad \beta = 1 + \frac{\mu - 3}{\mu - 2}. \tag{11}$$

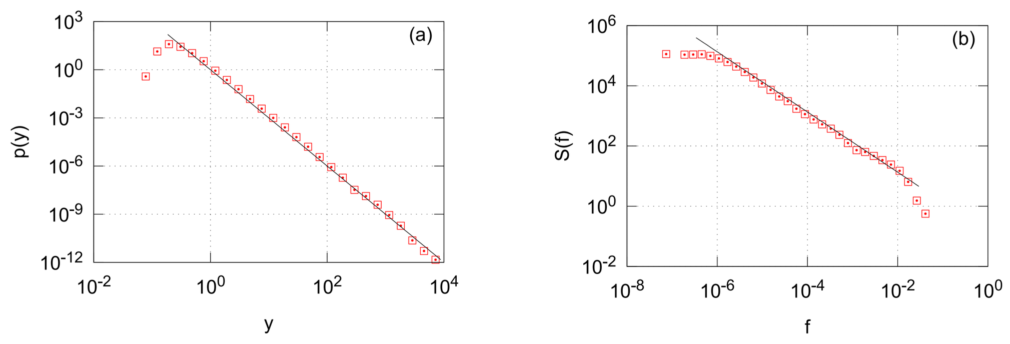

Figure 3: Statistical properties, PDF (a) and PSD (b), of numerically evaluated nonlinear GARCH process Eq. (7) with $\mu = 3$ (red squares). Black curves show power law approximations: (a) x^{-3} and (b) $1/f$. Used parameter set: $a = 10^{-6}$, $b = 10^{-3}$, $c = 1$. QR code will take you to interactive HTML5+Javascript app on Physics of Risk (http://mokslasplius.lt/rizikos-fizika) website.

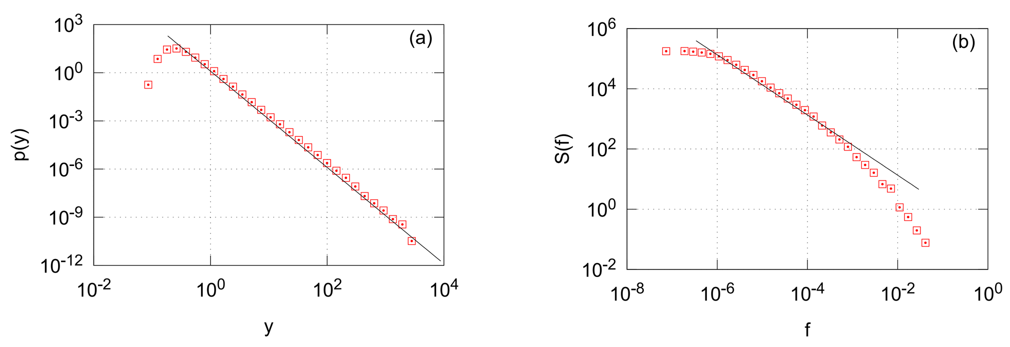

Figure 4: Statistical properties, PDF (a) and PSD (b), of numerically evaluated nonlinear GARCH process Eq. (8) with $\mu = 3$ (red squares). Black curves show power law approximations: (a) x^{-3} and (b) $1/f$. Used parameter set: $a = 10^{-6}$, $b = 10^{-3}$, $c = 2\sqrt{\frac{2}{\pi}} \cdot 10^{-3} \approx 1.595769 \cdot 10^{-3}$. QR code will take you to interactive HTML5+Javascript app on Physics of Risk (http://mokslasplius.lt/rizikos-fizika) website.

Note that in both case we have selected parameters as to set $C = 0$ and $C' = 0$. This is necessary as parameters C and C' determine if power-law behavior is possible to observe. If C and C' deviate from zero - extreme events rapidly become less probable to observe.

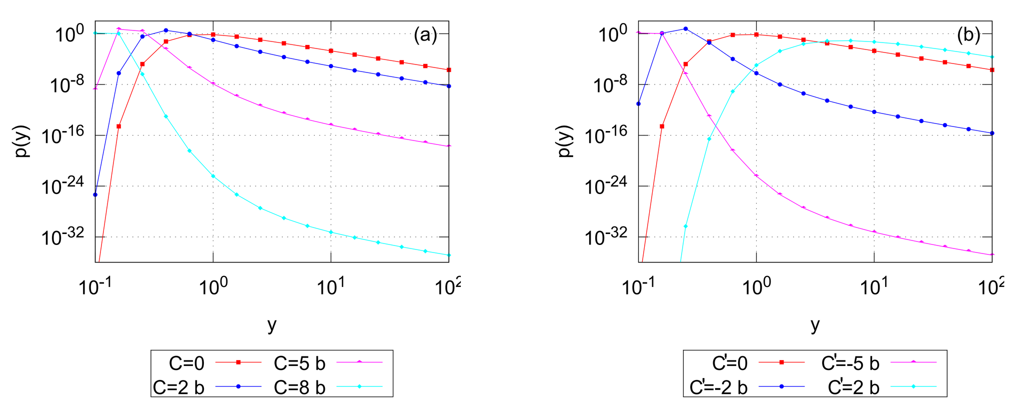

Figure 5: Analytical stationary PDF of (a) Eq. (7) and (b) Eq. (8) with non-zero values of C and C'. Used parameter set: $a = 10^{-6}$, $b = 10^{-3}$ and c values is set according to the parameter relations.

Conclusions

We have considered two possible nonlinear modifications of a GARCH(1,1) process and compared them to a well-known nonlinear SDE (1), which is able to reproduce long-range memory PSD (generating $1/f$ noise). Numerical evaluation of Eqs. (7) and (8) with suitably chosen parameters confirms the presence of a wide power-law region in the PSD of the resulting time series. Effectively this mean that we observe indications of long-range memory in memory-less model.

The results are especialy interesting as long-range memory ($1/f$ PSD) is considered to be one of the stylized facts of the financial markets as well as other complex systems. The obtained results and proposed nonlinear GARCH processes should be useful for creation and application of ARCH family models that correctly reproduce the PSD of the financial time series as well.

References

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