Voter model for finance, opinion dynamics, and census

Aleksejus Kononovicius

Institute of Theoretical Physics and Astronomy, Vilnius University

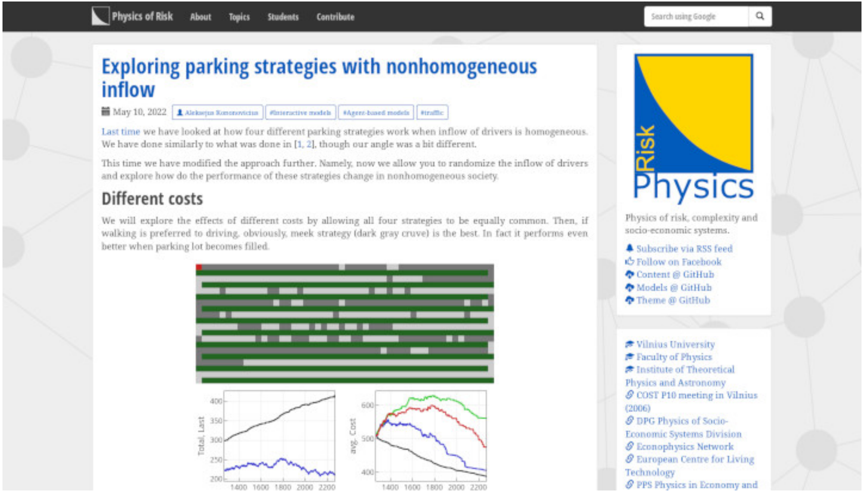
email: aleksejus.kononovicius@tfai.vu.lt
www: kononovicius.lt, rf.mokslasplius.lt

Complex Physical and Social Systems Group



Info: VU Faculty of Physics website. Photos from public Facebook pages.

Physics of Risk blog



Website: https://rf.mokslasplius.lt

Extensive and nonextensive voter models

Classic voter model

Originally defined similarly to a cellular automaton:

- Agents are the cells of a two dimensional grid.
- Each agent is in one of the two states: $+1$ or -1.
- During each time step:
  - agent (A) is selected,
  - its neighbor (B) is selected,
  - A copies the state of B.

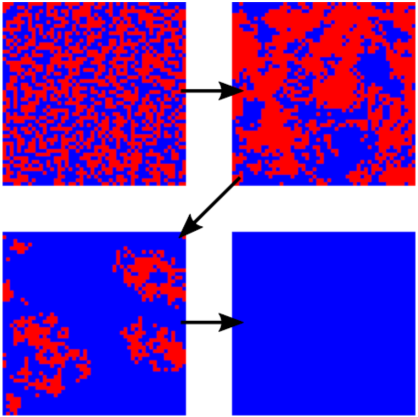
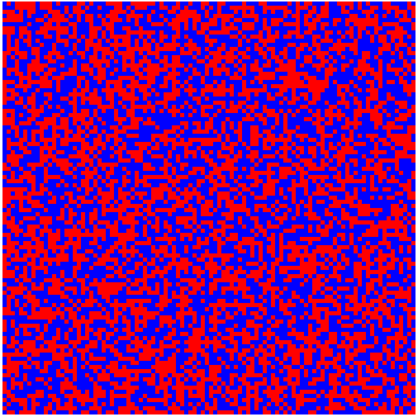


Fig.: Voter model (Physics of Risk)

Noisy voter model

- Social systems rarely reach a full consensus.
- To break the consensus lets include “thermal” noise. Namely, lets change our rule to:
  - During each time step:
    - agent (A) is selected,
    - with probability p A flips his state,
    - otherwise its neighbor (B) is selected
    - and A copies the state of B.



Voter model is one–step processes

- During each time step at most one spin changes.
- Lets use mean–field approximation, to get probabilities of the both possible changes:

$$P(X_+ \to X_+ + 1) = \frac{X_-}{N}\left[p + (1-p)\frac{X_+}{N}\right] = (1-x)\left[p + (1-p)x\right],$$

$$P(X_+ \to X_+ - 1) = \frac{X_+}{N}\left[p + (1-p)\frac{X_-}{N}\right] = x\left[p + (1-p)(1-x)\right],$$

here $x = \frac{X_+}{N}$.

Continuous limit of the noisy voter model

In continuous limit, any discrete one–step process of the following form:

$$P(X_+ \to X_+ \pm 1) = \lambda^{\pm}\Delta t,$$

is well approximated by the following stochastic differential equation:

$$\mathrm{d}x = \frac{\lambda^+ - \lambda^-}{N}\mathrm{d}t + \sqrt{\frac{\lambda^+ + \lambda^-}{N^2}}\mathrm{d}W.$$

For the noisy voter model (assuming $\Delta t = N^{-1}$), we get:

$$\mathrm{d}x = p(1 - 2x)\mathrm{d}t + \sqrt{\frac{1}{N}[\ldots]}\mathrm{d}W.$$

Thus it is an **extensive model**.

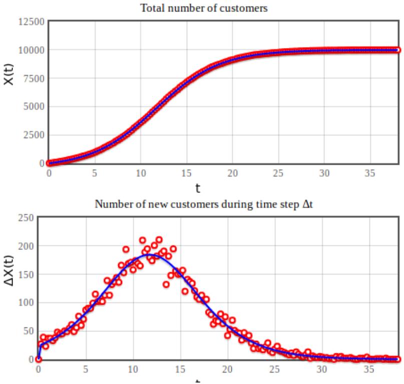
Bass diffusion model

- Consider consumers and potential consumers of a durable good:

$$P(X \to X-1) = 0.$$

- Then in continuous limit:

$$\mathrm{d}x \approx (1-x)\,[p + (1-p)x]\,\mathrm{d}t.$$



ABM vs ODE.

Fig.: Unidirectional Kirman's model (Physics of Risk)

Construction of the nonextensive voter model

- We need to make the imitation process N times more active.
- We can do it in framework we have been using until now, but the change is somewhat hard to conceptualize.
- So **lets change the framework! Instead of probabilities lets consider event rates.**
- We have only two possible events: birth and death in respect to X_+.
- Rates must be positive, but otherwise are unconstrained, so:

$$\lambda^+ = (N - X_+)\left[\varepsilon_+ + X_+\right], \quad \lambda^- = X_+\left[\varepsilon_- + (N - X_+)\right].$$

The nonextensive voter model was already known as Kirman's model.

Numerical simulation using event rates

Gillespie method (in general):

- Draw inter–event time τ_i from exponential distribution using total event rate λ.
- Update clock: $t_{i+1} = t_i + \tau_i$.
- With $p^{(k)} = \frac{\lambda^{(k)}}{\lambda}$ execute event k.
- Update total event rate $\lambda = \sum_k \lambda^{(k)}$.

In voter models we have just two events:

- with rate λ^+ we $X_+ \leftarrow X_+ + 1$,
- with rate λ^- we $X_+ \leftarrow X_+ - 1$.

[Gillespie (2007)]

Continuous limit for the nonextensive voter model

We know that:

$$\mathrm{d}x = \frac{\lambda^+ - \lambda^-}{N}\mathrm{d}t + \sqrt{\frac{\lambda^+ + \lambda^-}{N^2}}\mathrm{d}W.$$

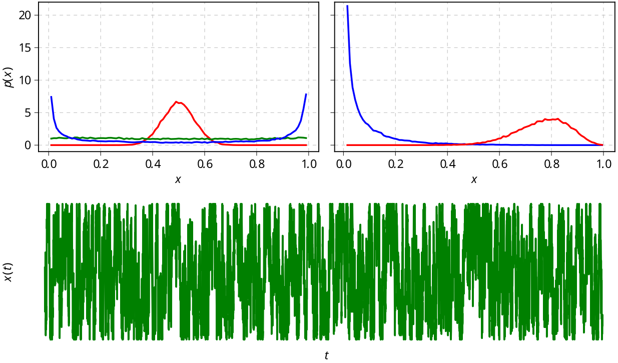
For the nonextensive model we have:

$$\mathrm{d}x \approx \left[\varepsilon_+ (1-x) - \varepsilon_- x\right]\mathrm{d}t + \sqrt{2x(1-x)}\mathrm{d}W.$$

From this it can be easily shown that:

$$\boxed{x \sim \mathcal{B}e(\varepsilon_+, \varepsilon_-)}.$$

Lets check numerically



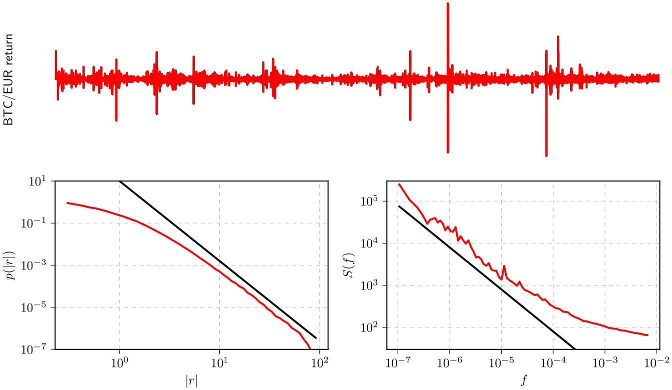
(left) $\varepsilon_+ = \varepsilon_- = \{0.01, 1, 100\}$.

(right) $\varepsilon_+ = \{0.2, 16\}$, $\varepsilon_- = 5$.

(bottom) Series with $\varepsilon_+ = \varepsilon_- = 1$.

Long–range memory in financial markets

Example: BTC time series



Return:

$$r = \ln P\,(t + \Delta t) - \ln P(t).$$

Empirical properties:

$$p(|r|) \propto r^{-4}, \quad S(f) \propto \frac{1}{f}.$$

Data: bitcoincharts.com.

Opinions as trading strategies

Noise traders:

$$D_c = r_0 N_c \xi .$$

Fundamentalists:

$$D_f = N_f \left(\ln P_f - \ln P \right) .$$

Equilibrium price:

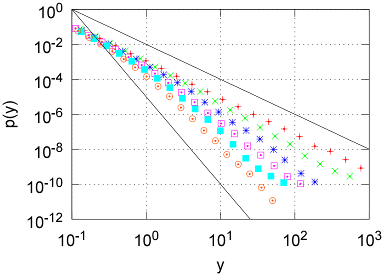
$$D_f + D_c = 0, \ \Rightarrow \ P = P_f \cdot \exp\left(r_0 \frac{N_c}{N_f} \xi \right) .$$

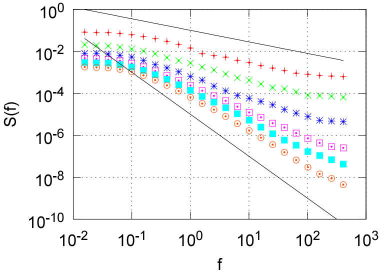
If $P_f = \mathrm{const}$ and $\xi(t)$ is fast, then return:

$$r \approx r_0 \frac{N_c}{N_f} \Delta\xi = r_0 \cdot \boxed{\frac{x}{1-x}} \cdot \Delta\xi$$

Fig.: Jeff Parker (caglecartoons).

Statistical properties of long–term component





Let $y = \frac{x}{1-x}$, then SDE:

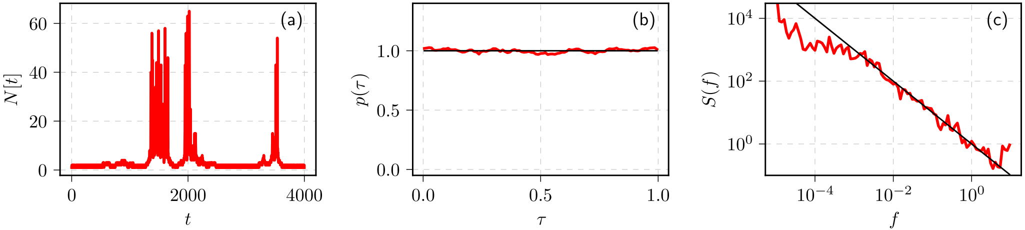
$$\mathrm{d}y \approx (2 - \varepsilon_2)\, y^{2+\alpha} \mathrm{d}t + \sqrt{2 y^{3+\alpha}} \mathrm{d}W.$$

Statistical properties:

$$p(y) \propto y^{-\varepsilon_2 - \alpha - 1}, \quad S(f) \propto f^{-1-\frac{\varepsilon_2+\alpha-2}{1+\alpha}}.$$

[Kononovicius *et al.*(2012), (2019)].

General class of nonlinear SDEs



Inter–event times:

$$\tau_{i+1} = \tau_i + \sigma\zeta_i,$$

with $\zeta_i \sim \mathcal{N}(0, 1)$.

Intensity $n = \frac{1}{\tau}$:

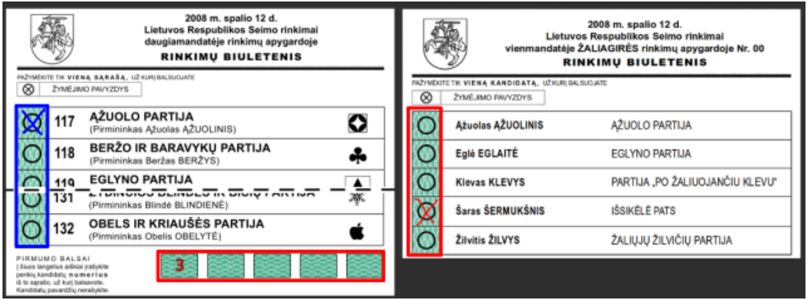
$$\mathrm{d}n = \sigma^2 n^4 \mathrm{d}t + \sigma n^{5/2} \mathrm{d}W,$$

here W is Wiener process.

[Kazakevičius *et al.*(2022)]

Nonextensive fluctuations in public opinion

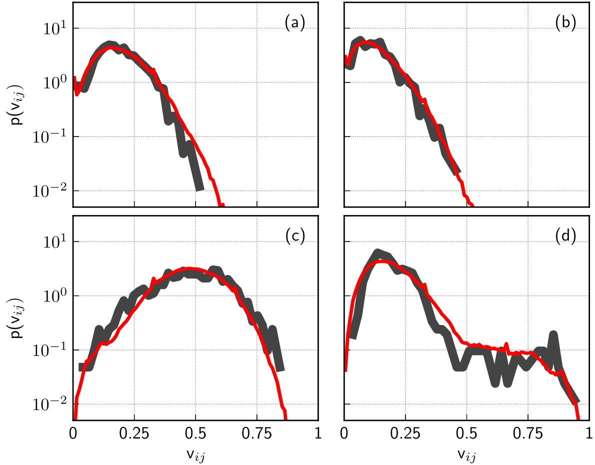
Extracting popular vote data



- Two–tier system.
- Held each 4 years.
- Polling station level data.

Fig.: Central Electoral Commission. Data: CEC (raw, Lithuanian), GitHub (processed, English)

PDFs over polling stations (1992)



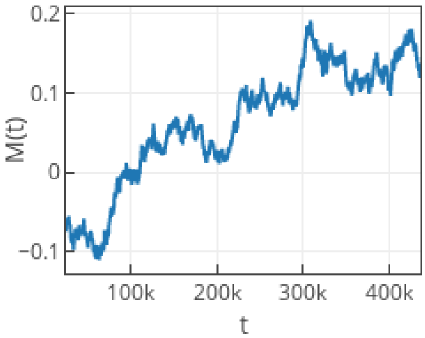
(a) "Sąjūdžio koalicija" (21%)
(b) LKDP (13%)
(c) LDDP (44%)
(d) others combined (22%)

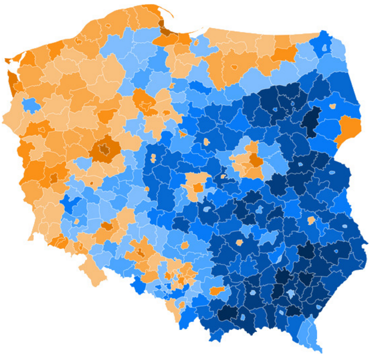
Mixture of two simulations.

[Kononovicius (2017)]

What is wrong with such approach

Model is temporal, data is (mostly) spatial.





Figs.: q-Voter model (Physics of Risk), Teratornis06@Wiki

But what if spatial units are inter–dependent?

PHYSICAL REVIEW LETTERS

Highlights Recent Accepted Collections Authors Referees Search Press About

Featured in Physics Editors' Suggestion Go Mobile »

Is the Voter Model a Model for Voters?

Juan Fernández-Gracia, Krzysztof Suchecki, José J. Ramasco, Maxi San Miguel, and Víctor M. Eguíluz

Phys. Rev. Lett. **112**, 158701 – Published 18 April 2014; Erratum Phys. Rev. Lett. **113**, 089903 (2014)

Physics See Focus story: Voter Model Works for US Elections

33

More

Article | References | Citing Articles (62) | Supplemental Material | PDF | HTML | Export Citation

ABSTRACT

The voter model has been studied extensively as a paradigmatic opinion dynamics model. However, its ability to model real opinion dynamics has not been addressed. We introduce a noisy voter model (accounting for social influence) with recurrent mobility of agents (as a proxy for social context), where the spatial and population diversity are taken as inputs to the model. We show that the dynamics can be described as a noisy diffusive process that contains the proper anisotropic coupling topology given by population and mobility heterogeneity. The model captures statistical features of U.S. presidential elections as the stationary vote-share fluctuations across counties and the long-range spatial correlations that decay logarithmically with the distance. Furthermore, it recovers the behavior of these properties when the geographical space is coarse grained at different scales—from the county level through congressional districts, and up to states. Finally, we analyze the role of the mobility range and the randomness in decision making, which are consistent with the empirical observations.

Issue

Vol. 112, Iss. 15 — 18 April 2014

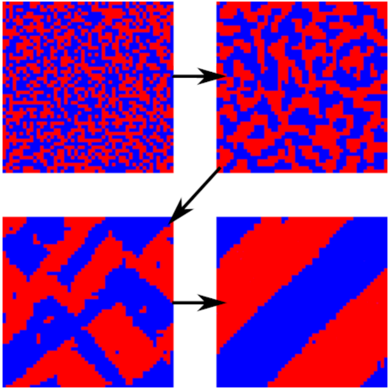
Reuse & Permissions

[Fernandez–Gracia *et al.* (2014)]

Kawasaki dynamics of Ising model

During each time step:

- pick particle (A),
- pick its neighbor (B),
- A and B both flip according to the usual rule.



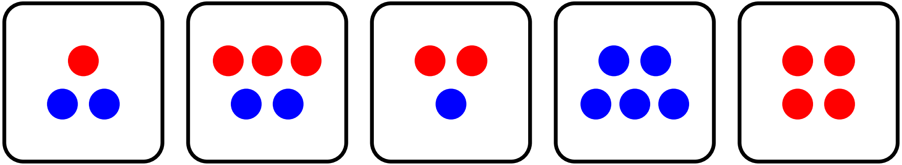
$T \ll T_c$.

Fig.: Kawasaki Ising model (Physics of Risk). [Kawasaki (1966), (1966)].

Setup of “Kawasaki” voter model

- Let there be N agents.
- Let there be T agent types.
- Let the types be fixed.
- Let agents reside in M compartments.
- Let capacity of each district be C.

N=20, T=2, M=5, C=5



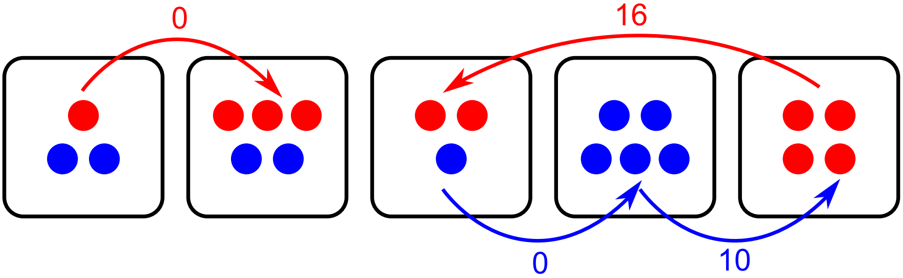
[Kononovicius (2019)]

Evolution of the “Kawasaki” voter model

Let the migration rate between districts be ($i \to j$ for type k):

$$\lambda_{(k)}^{i\to j} = \begin{cases} X_i^{(k)}\left(\varepsilon^{(k)} + X_j^{(k)}\right) & \text{if } i \neq j \text{ and } N_j < C, \\ 0 & \text{otherwise,} \end{cases}$$

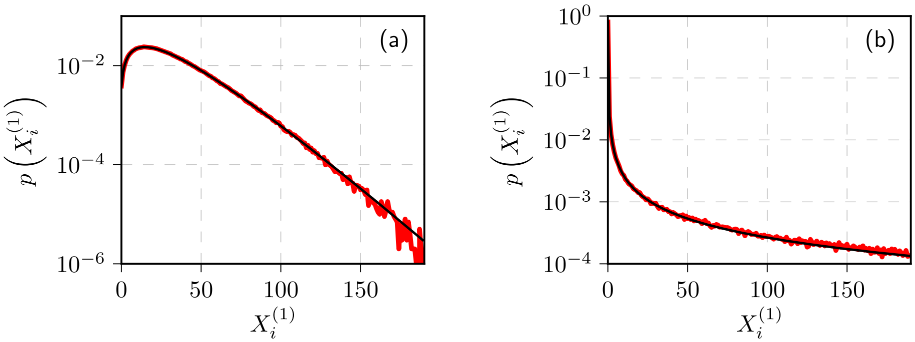
N=20, T=2, M=5, C=5, ε=2



What can be done analyticaly?

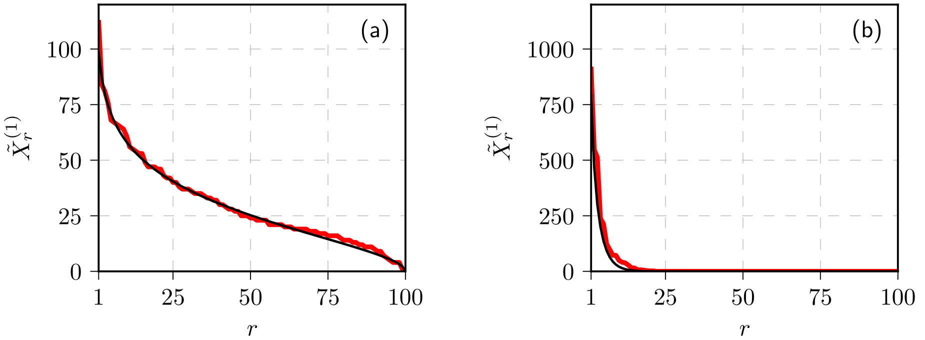
- If the capacity is effectively infinite $C = N$, we know closed form expressions for the total entry/exit rates. **Thus we can get the closed form expression** for the stationary distribution of $X_i^{(k)}$ for fixed i and k.
- If capacity is finite, we have to use detailed balance to get the stationary distribution. **This works, but scales poorly**.
- The problem is that we are more interested in the compartmental distribution of $f_i^{(k)} = X_i^{(k)}/N_i$. **This is seems impossible**.

PDF of $X_i^{(k)}$ for the infinite capacity



Model (red) vs Beta–fit (black): $N = 3000$, $T = 1$, $M = 100$ and $C = N$ (both), $\varepsilon^{(1)} = 2$ (a) and 0.03 (b).

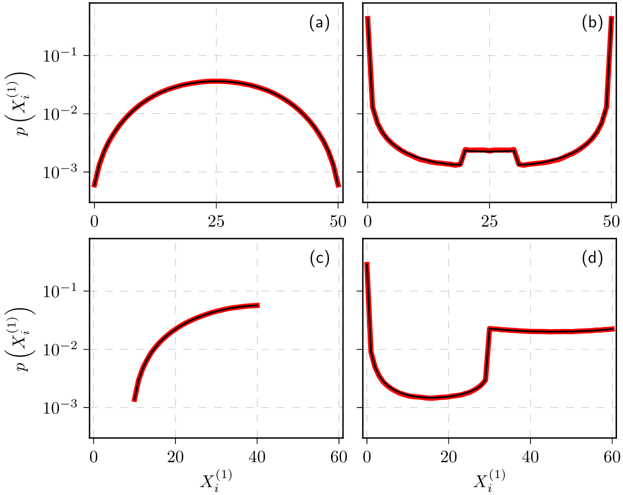
CRSD of $X_i^{(k)}$ for the infinite capacity



Model (red) vs Beta–fit (black): $N = 3000$, $T = 1$, $M = 100$ and $C = N$ (both), $\varepsilon^{(1)} = 2$ (a) and 0.03 (b).

CRSD – abbr. compartmental rank–size distribution.

PDF of $X_i^{(k)}$ for the finite capacity



(a) $N = 100$, $M = 2$, $T = 2$, $\varepsilon = 2$, $C = 60$

(b) $N = 100$, $M = 2$, $T = 2$, $\varepsilon = 0.03$, $C = 80$

(c) $N = 90$, $M = 3$, $T = 1$, $\varepsilon = 2$, $C = 40$

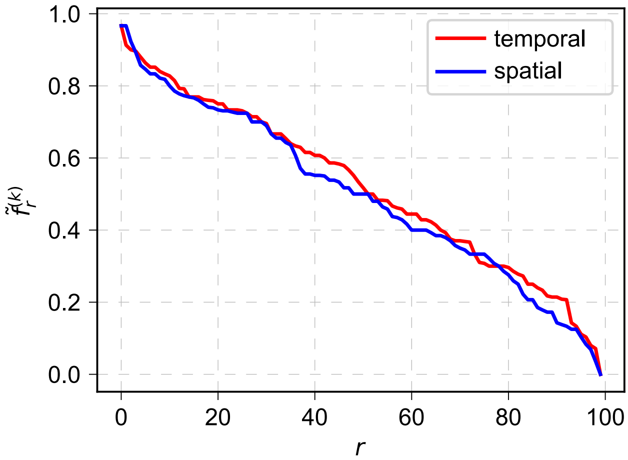
(d) $N = 90$, $M = 3$, $T = 1$, $\varepsilon = 0.03$, $C = 60$

Spatio–temporal (RSD) symmetry

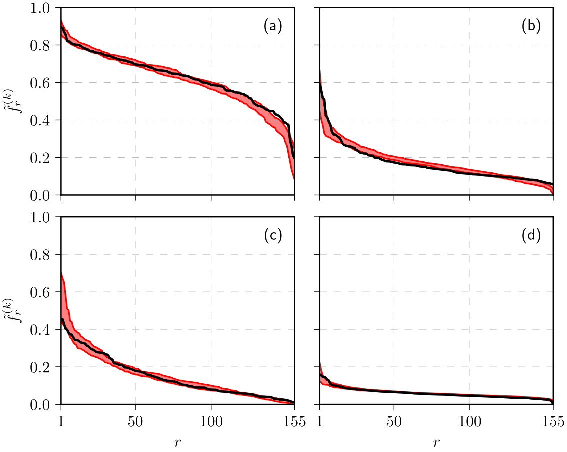
We observe:

$$f_i^{(k)} = \frac{X_i^{(k)}}{N_i}.$$

Model parameters:
$N = 2600$, $T = 2$, $M = 100$,
$C = 30$ and $\varepsilon = 2$.



Application: Ethnic groups in London (UK 2011)



(a) White: $N^{(w)} = 48515$, $\varepsilon^{(w)} = 2.5$.

(b) Asian: $N^{(a)} = 12865$, $\varepsilon^{(a)} = 4$.

(c) Black: $N^{(b)} = 11470$, $\varepsilon^{(b)} = 1.5$,

(d) Other: $N^{(o)} = 4495$, $\varepsilon^{(o)} = 15$.

Other parameters: $N = 77345$, $M = 155$, $C = 600$.

Red areas show 95% confidence intervals for the model.

Data: Office for National Statistics (UK).

Summary

Some key ideas

- Voter models are not only for the voters.
- Nonlinearity “remembers”.
- Nonextensive voter model can encode heterogeneity.
- Opinion dynamics is often more alike Kawasaki than Glauber dynamics.

Foreground: (source lost). Background: “spinsons” used in numerous papers by a sociophysics group based in Wroclaw.

Thank you!

email: aleksejus.kononovicius@tfai.vu.lt
www: kononovicius.lt, rf.mokslasplius.lt

Faculty of Physics