Non–extensive voter model for socio–economic phenomena

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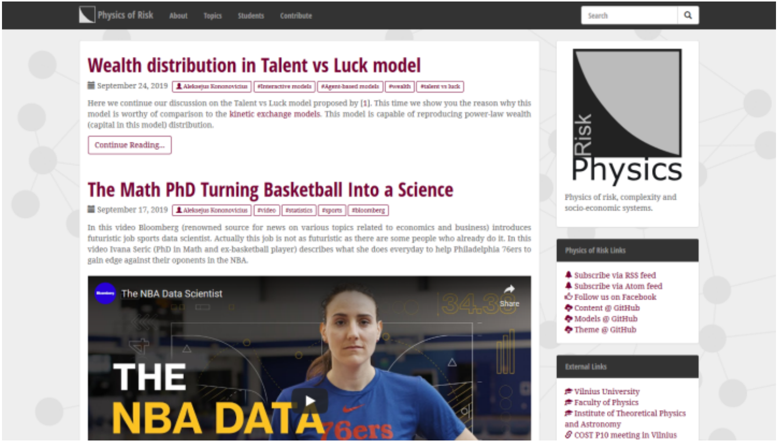
Physics of Risk



Blogging since 2006: $\sim$ 100 interactive models, $\sim$ 170 texts w/o models.

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What this talk is about?

- Real life socio–economic systems are finite, yet their statistical properties are size (N) independent.
- Agent–based models are also finite, yet often the statistical properties of ABMs are N-dependent.

- Extensive models exhibit no fluctuations or only if N is finite.
- Non–extensive models exhibit fluctuations even if $N \to \infty$.

Agents we will (left) and will not (right) talk about.

Note: I use terms “extensive” and “non–extensive” rather loosely.

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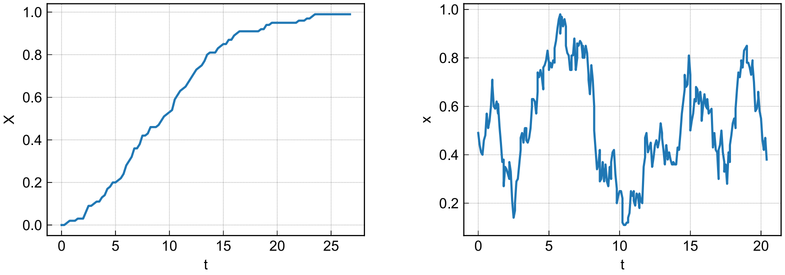
Extensive vs non–extensive



Typical time series of an extensive model (left) and a non–extensive model (right).

Figs.: http://rf.mokslasplius.lt/unidirectional-kirman-model/, http://rf.mokslasplius.lt/kirman-ants/.

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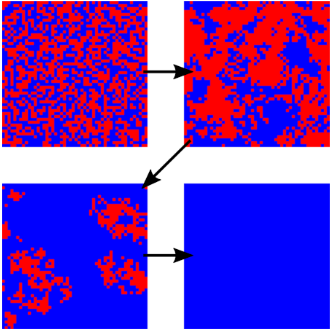
Extensive and non–extensive voter models

Classic voter model

Originally defined similarly to a cellular automaton:

- Agents are the cells of a two dimensional grid.
- Each agent is in one of the two states: $+1$ or -1.
- During each time step:
  - agent (A) is selected,
  - its neighbor (B) is selected,
  - A copies the state of B.



Consensus emerges in the classic voter model.

Collage from: http://rf.mokslasplius.lt/voter-model/

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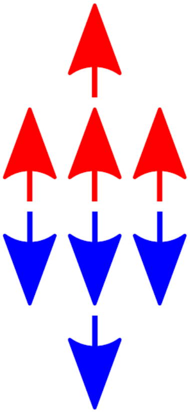
Zero temperature Ising model

The Ising model is quite similar in formulation:

- Particles are placed on a grid.
- Each particle has a spin of $+1$ or -1.
- During each time step:
  - a particle is selected,
  - it changes the state to align with its neighbors
  - or flips randomly if there is no clear majority.



This state is stable for the Ising model, but unstable for the voter model.

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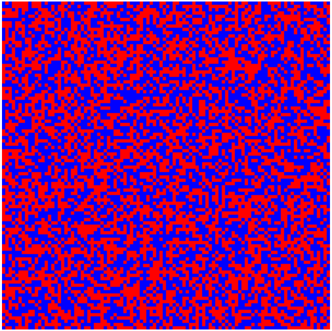
Noisy voter model

- Social systems rarely reach a full consensus.
- To break the consensus lets include "thermal" noise. Namely, lets change our rule to:
  - During each time step:
    - agent (A) is selected,
    - with p_r A flips his state,
    - otherwise its neighbor (B) is selected
    - and A copies the state of B.



No consensus ($p_r = 0.05$).

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Voter models as one–step processes

- During each time step at most one spin changes.
- Lets use mean–field approximation,
- to get probabilities of the two possible changes:

$$P(X_+ \rightarrow X_+ + 1) = \frac{X_-}{N}\left[p_r + (1-p_r)\frac{X_+}{N}\right] = (1-x)\left[p_r + (1-p_r)x\right],$$

$$P(X_+ \rightarrow X_+ - 1) = \frac{X_+}{N}\left[p_r + (1-p_r)\frac{X_-}{N}\right] = x\left[p_r + (1-p_r)(1-x)\right],$$

here $x = \frac{X_+}{N}$.

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Continuous limit of the noisy voter model

In continuous limit, any discrete one–step process of the following form:

$$P(X_+ \to X_+ \pm 1) = \lambda^{\pm} \Delta t,$$

is well approximated by the following stochastic differential equation:

$$\mathrm{d}x = \frac{\lambda^+ - \lambda^-}{N} \mathrm{d}t + \sqrt{\frac{\lambda^+ + \lambda^-}{N^2}} \mathrm{d}W.$$

For the noisy voter model (assuming $\Delta t = N^{-1}$), we get:

$$\mathrm{d}x = p_r (1 - 2x) \mathrm{d}t + \sqrt{\frac{1}{N} \left[2x(1-x)(1-p_r) + p_r\right]} \mathrm{d}W.$$

Thus it is an **extensive model**.

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Bass diffusion model

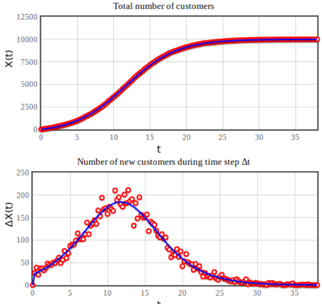
as a special case of the extensive voter model

- We consider consumers ($+1$) and potential consumers (-1) of a durable good.
- Forbid $+1 \rightarrow -1$:

$$P(X_+ \rightarrow X_+ - 1) = 0.$$

- Then in continuous limit:

$$\mathrm{d}x \approx (1-x)\left[p_r + (1-p_r)x\right]\mathrm{d}t.$$



"Unidirectional" voter model vs Bass diffusion ODE.

Fig.: http://rf.mokslasplius.lt/unidirectional-kirman-model/

Constructing the non–extensive voter model

- We need to make the imitation process N times more active.
- We can do it in framework we have been using until now, but the change is somewhat hard to conceptualize.
- So **lets change the framework! Instead of probabilities lets consider event rates.**
- We have only two possible events: generation (birth) and recombination (death) in respect to X_+.
- Rates must be positive, but otherwise are unconstrained, so:

$$\lambda^+ = (N - X_+)\left[\varepsilon_+ + X_+\right], \quad \lambda^- = X_+\left[\varepsilon_- + (N - X_+)\right].$$

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Numerical simulation using event rates

Gillespie method (in general):

- Draw inter–event time τ_i from exponential distribution using total event rate λ.
- Update clock: $t_{i+1} = t_i + \tau_i$.
- With $p^{(k)} = \frac{\lambda^{(k)}}{\lambda}$ execute event k.
- Update total event rate $\lambda = \sum_k \lambda^{(k)}$.

In voter models we have just two events:

- with rate λ^{+} we increment X_{+},
- with rate λ^{-} we decrement X_{+}.

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Continuous limit for the non–extensive voter model

We know that:

$$dx = \frac{\lambda^+ - \lambda^-}{N} dt + \sqrt{\frac{\lambda^+ + \lambda^-}{N^2}} dW.$$

For the non–extensive model we have:

$$dx \approx \left[\varepsilon_+ \left(1 - x\right) - \varepsilon_- x\right] dt + \sqrt{2x(1-x)} dW.$$

From this it can be easily shown that:

$$\boxed{x \sim \mathcal{B}e(\varepsilon_+, \varepsilon_-)}.$$

The non–extensive voter model is also known as the Kirman model.

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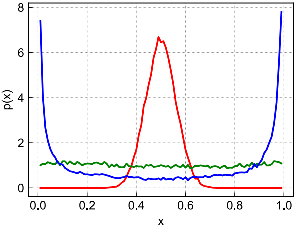
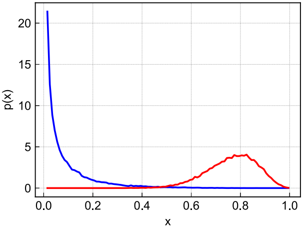
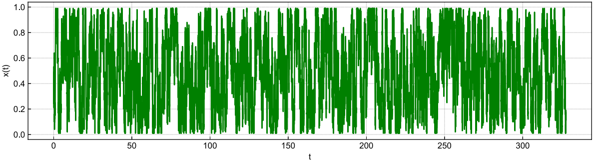
Lets see numerically



$\varepsilon_+ = \varepsilon_- = \{0.01, 1, 100\}$



$\varepsilon_+ = \{0.2, 16\}, \varepsilon_- = 5$



Typical series with $\varepsilon_+ = \varepsilon_- = 1$.

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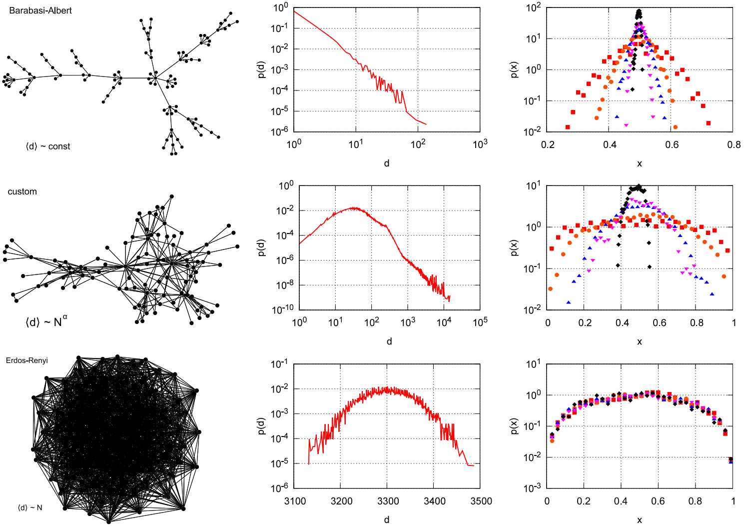
Topology does matter!



Kononovicius & Ruseckas, EPJ B 87: 169 (2014).

Financial markets

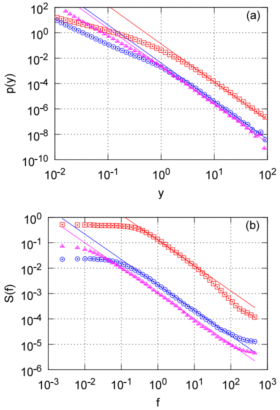
Fat–tails and long–range memory

- Let the two states represent chartist and fundamentalist traders.
- Let Walrass equilibrium hold.
- Then ignoring short–term fluctuations, the return is driven by:

$$y = \frac{x}{1-x}.$$

- Which is described by the following SDE:

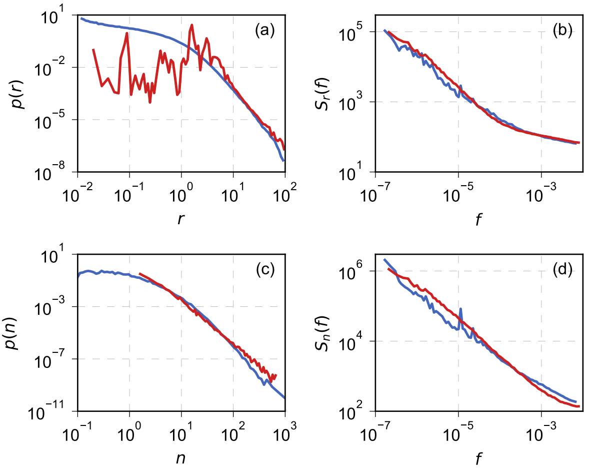
$$\mathrm{d}y \approx (2-\varepsilon_{cf})\, y^{2+\alpha} \mathrm{d}t_s + \sqrt{2y^{3+\alpha}} \mathrm{d}W.$$



$$p(y) \sim y^{-3},\ S(f) \sim 1/f.$$

Kononovicius & Gontis, Physica A 391: 1309-1314 (2012); and many later works

Latest paper: Combined order–book model



BTC stats (red) vs model stats (blue).

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Kononovicius & Ruseckas, Physica A 525: 171-191 (2019); relies on: Kanazawa *et al.*, PRL 120: 138301 (2018).

Lithuanian parliamentary elections

Based on: Kononovicius, Complexity 2017: 7354642 (2017).

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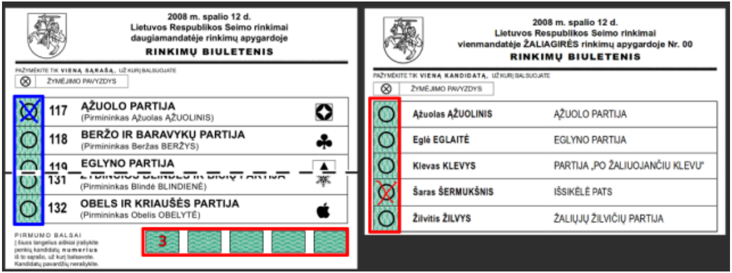
Example ballots



We are interested in political party performance (blue).

- Two–tier system.
- Held each 4 years.
- Polling station level data ($\sim$ 2000 data points).

Fig.: Central Electoral Commission

Data sources: CEC (full, in Lithuanian), GitHub (partial, in English)

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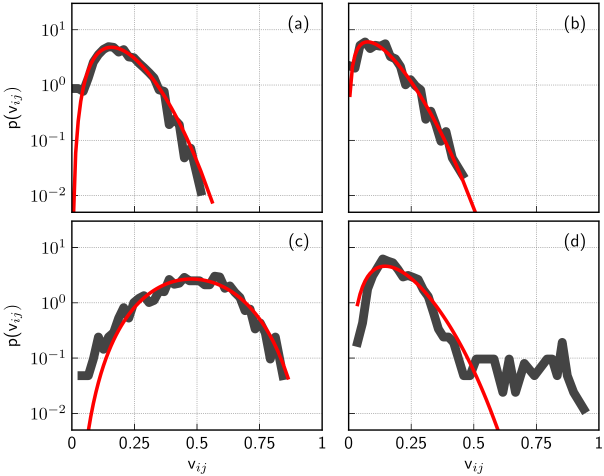
PDFs over polling stations (1992)



Three main parties ((a)-(c)) and other parties (d): empirical data (gray) vs Beta–fit (red).

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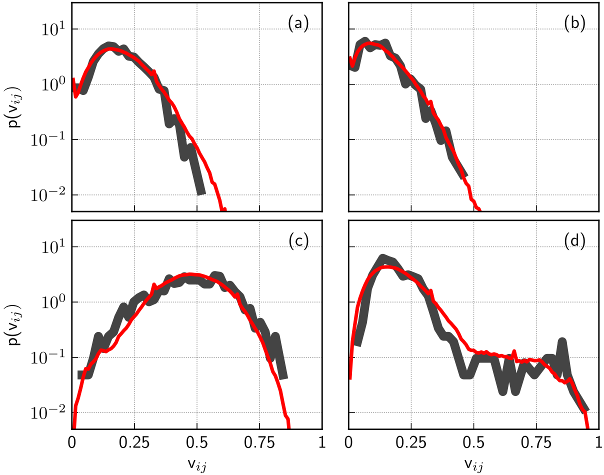
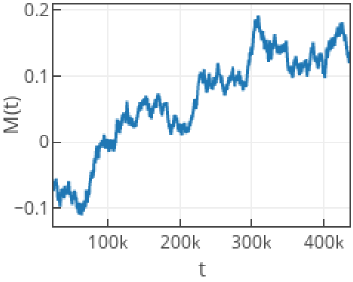
Multi–state non–extensive voter model (1992)

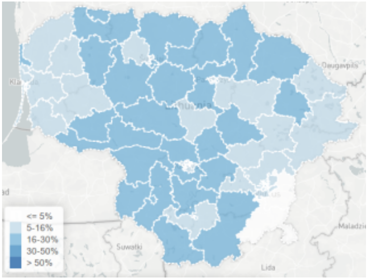


Three main parties ((a)-(c)) and other parties (d): empirical data (gray) vs the model (red).

What is wrong with this approach?

- Models are temporal, space is abstracted away.
- Data is spatial, and observe rarely in time.





Figs.: http://rf.mokslasplius.lt/q-voter-model/, http://rinkimurezultatai.lt

Independence assumption made in: (Kononovicius, 2017), (Sano *et al.*, 2017), (Braha & de Aguiar, 2017), (Fenner *et al.*, 2017)

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But what if spatial units are inter–dependent?

Complicated approach building on commuting patterns

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Is the Voter Model a Model for Voters?

Juan Fernández-Gracia, Krzysztof Suchecki, José J. Ramasco, Maxi San Miguel, and Víctor M. Eguíluz

Phys. Rev. Lett. **112**, 158701 – Published 18 April 2014; Erratum Phys. Rev. Lett. **113**, 089903 (2014)

Physics See Focus story: Voter Model Works for US Elections

33

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ABSTRACT

The voter model has been studied extensively as a paradigmatic opinion dynamics model. However, its ability to model real opinion dynamics has not been addressed. We introduce a noisy voter model (accounting for social influence) with recurrent mobility of agents (as a proxy for social context), where the spatial and population diversity are taken as inputs to the model. We show that the dynamics can be described as a noisy diffusive process that contains the proper anisotropic coupling topology given by population and mobility heterogeneity. The model captures statistical features of U.S. presidential elections as the stationary vote-share fluctuations across counties and the long-range spatial correlations that decay logarithmically with the distance. Furthermore, it recovers the behavior of these properties when the geographical space is coarse grained at different scales—from the county level through congressional districts, and up to states. Finally, we analyze the role of the mobility range and the randomness in decision making, which are consistent with the empirical observations.

Issue

Vol. 112, Iss. 15 — 18 April 2014

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Fernandez–Garcia *et al.*, PRL 112: 158701 (2014)

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Compartmental voter model

The voter model, which is not a model for voters

Based on: Kononovicius, accepted to J. Stat. Mech., arXiv: 1906.01842 [physics.soc-ph]

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Socio–demographics impact voting behavior

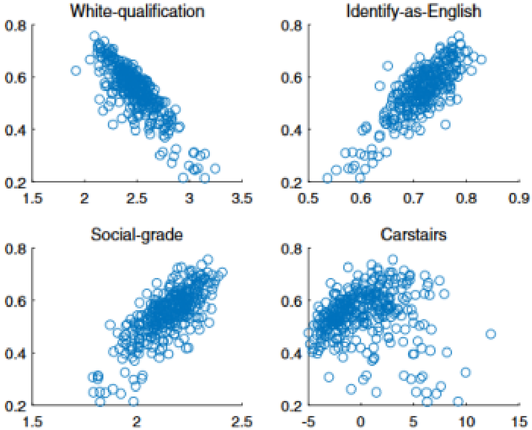

Fig. 2. (Color online) Scatter plots for the four chosen covariates; the y-values represent the proportion of Leave votes for a district and the x-values represent the values of the covariate for the district.

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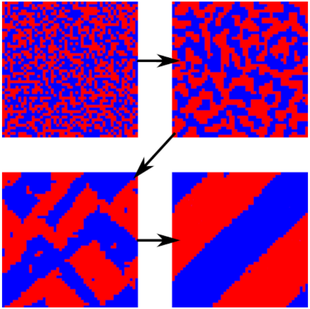
Fenner *et al.*, Int J Mod Phys C 28: 1750132 (2017)

Kawasaki dynamics of Ising model

During each time step:

- pick particle (A),
- pick its neighbor (B),
- A and B switch places based on how likely both states are according to the Boltzmann distribution.

In physical sense we are modeling transport phenomena and ignoring magnetization itself.



Kawasaki dynamics in Ising model with $T \ll T_c$.

Fig.: unpublished post on Physics of Risk

Kawasaki, Phys. Rev. 145 (1966) and Phys. Rev. 148 (1966)

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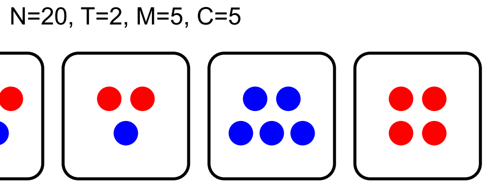
Statics of the compartmental voter model

- Let there be N agents (people).
- Let each agent be one of T types (splits in socio–demographic category).
- Let those types be fixed.
- Let agents reside in M compartments (any real spatial unit).
- Let capacity of each district be C.



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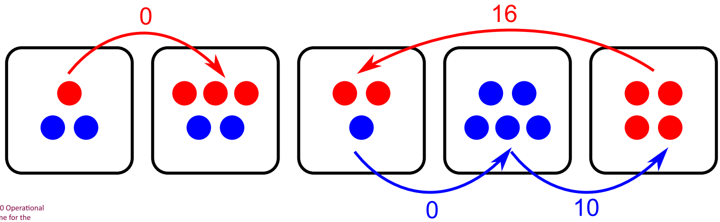
Dynamics in the compartmental voter model

Let the migration rate between districts be ($i \to j$ for type k):

$$\lambda_{(k)}^{i \to j} = \begin{cases} X_i^{(k)} \left(\varepsilon^{(k)} + X_j^{(k)} \right) & \text{if } i \neq j \text{ and } N_j < C, \\ 0 & \text{otherwise,} \end{cases}$$

in the above $X_i^{(k)}$ is the number of type k agents in district i, $\varepsilon^{(k)}$ is individual transition rate for type k, N_j is total number of agents in district j.

N=20, T=2, M=5, C=5, ε=2



What can be done analyticaly?

- If capacity is infinite $C = N$, we know closed form expressions for the total entry/exit rates. **Thus we can get closed form expression** for the stationary distribution of $X_i^{(k)}$ for fixed i and k.
- If capacity is finite, we have to use detailed balance to get the stationary distribution. **This works, but scales badly**.
- The problem is that we are more interested in the compartmental distribution of $f_i^{(k)} = X_i^{(k)}/N_i$. **This is likely impossible**.

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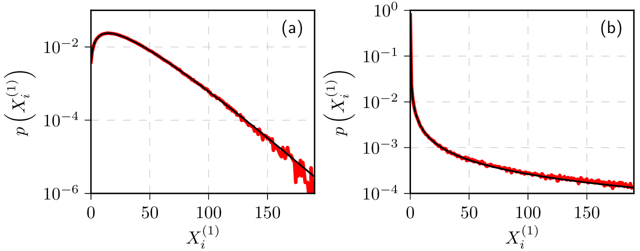
PDF of $X_i^{(k)}$ for the infinite capacity



Model (red) vs Beta–fit (black): $N = 3000$, $T = 1$, $M = 100$ and $C = N$ (both), $\varepsilon^{(1)} = 2$ (a) and 0.03 (b).

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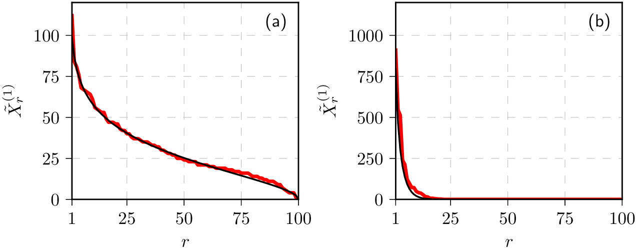
CRSD of $X_i^{(k)}$ for the infinite capacity



Model (red) vs Beta–fit (black): $N = 3000$, $T = 1$, $M = 100$ and $C = N$ (both), $\varepsilon^{(1)} = 2$ (a) and 0.03 (b).

CRSD – abbr. compartmental rank–size distribution at fixed time.

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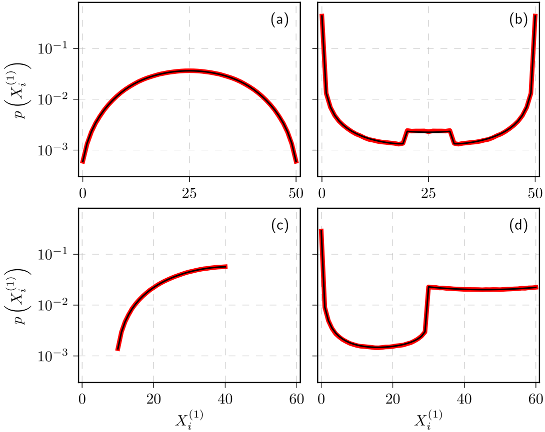
PDF of $X_i^{(k)}$ for the finite capacity



$N = 100$, $M = 2$ and $T = 2$ ((a) and (b)), $N = 90$, $M = 3$ and $T = 1$ ((c) and (d)), $C = 40$ (c), 60 ((a) and (d)) and 80 (b), $\varepsilon = 2$ ((a) and (c)) and 0.03 ((b) and (d)).

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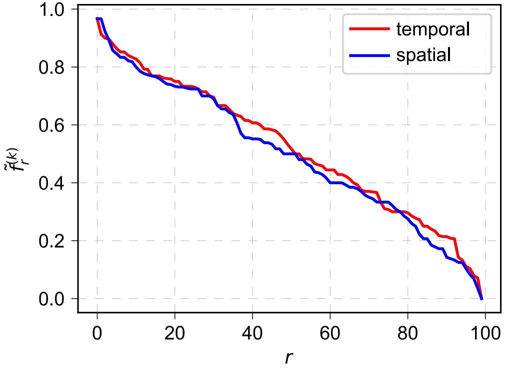
Spatio–temporal (RSD) symmetry



$N = 2600$, $T = 2$, $M = 100$, $C = 30$ and $\varepsilon = 2$

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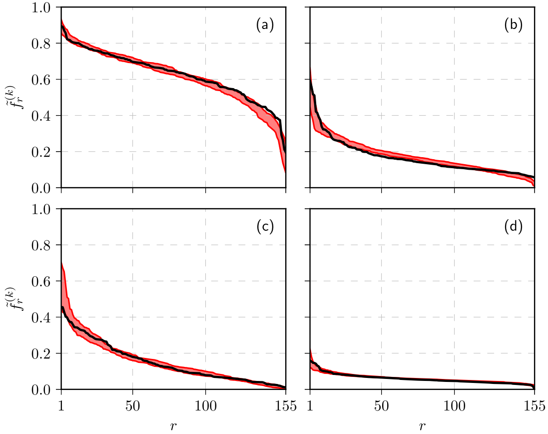
UK census 2011: Ethnic groups in London



Considered groups (black curves): (a) White, (b) Asian, (c) Black, (d) other. Model (red areas): $N^{(w)} = 48515$, $N^{(a)} = 12865$, $N^{(b)} = 11470$ and $N^{(o)} = 4495$ ($N = 77345$), $\varepsilon^{(w)} = 2.5$, $\varepsilon^{(a)} = 4$, $\varepsilon^{(b)} = 1.5$, $\varepsilon^{(o)} = 15$, $M = 155$, $C = 600$.

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Red areas show 95% confidence intervals for the model.

Summary

Some key ideas

- Voter models are not only for the voters.
- Non–extensive voter model can quite nicely reflect social heterogeneity.
- You can remember, even if you have no memory. Non–linearity does the job for you.
- Typical opinion dynamics models correspond to Glauber dynamics, the data is more alike Kawasaki dynamics, but this difference doesn't seem to matter much.
- Collectively we are all curious spinsons.

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Thank you!

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www: http://kononovicius.lt/, http://rf.mokslasplius.lt/