Double stochastic model of return in financial markets

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Abstract

Small scale, of order comparable with minutes or hours, time series drawn from empirical financial market data yield sophisticated statistical properties. What is the most fascinating is that many of these, in classical sense, anomalous features appear to be universal. Analysis of vast amounts of empirical data from around the world have helped to establish a variety of so-called stylized facts [1, 2, 3], which can be seen as statistical signatures of various financial processes. In this poster we consider price evolution process - i.e. modeling of return, which relates towards aforementioned external observable of financial markets as

$$r_\tau(t) = \ln\left[\frac{\pi(t)}{\pi(t-\tau)}\right], \tag{1}$$

here $\pi(t)$ is price function of time and τ is time scale of return. It is known that the probability density function (further in this poster abbreviated as PDF) of the return is successfully generalized within a non-extensive statistical framework [4]. The return has a distribution that is very well fitted by q-Gaussians, only slowly becoming Gaussian as the time scale approaches months, years and larger time scales.

Tackling modeling of the return we have started with the class of nonlinear stochastic differential equations (recently generalized in [5]),

$$\mathrm{d}x = \left(\eta - \frac{\lambda}{2}\right) x^{2\eta-1}\mathrm{d}t + x^{\eta}\mathrm{d}W, \tag{2}$$

whose time series have q-Gaussian stationary distribution and $1/f^\beta$ (here exponent $\beta = 1 + \frac{\lambda-3}{2(\eta-1)}$) power spectral density (further abbreviated as PSD). Using this able class of stochastic differential equations we have proposed a double stochastic process driven by the nonlinear scaled stochastic differential equation and q-Gaussian noise. Time series of the proposed process yield dynamical and stationary statistical properties resembling those of actual financial markets [6, 7].

Stochastic model with q-Gaussian PDF and $1/f^\beta$ PSD

It is known that if drift, $A(x)$, and diffusion, $B(x)$, functions of SDE doesn't depend on time then stationary PDF, $p(x)$, can be expressed trough those functions [8]. Thus we can express one of those functions trough the PDF and another function,

$$A(x) = \frac{1}{2}B^2(x)\frac{\partial_x p(x)}{p(x)} + B(x)\partial_x B(x). \tag{3}$$

In [6] we express q-Gaussian for dimensionless variable, $x = \frac{r}{r_0}$, in more transparent form than original - $p(x) = C(1+x^2)^{-\lambda/2}$ (here C is normalization constant). And by setting $B(x) = (1+x^2)^{\frac{\eta}{2}}$, this selection eliminates steady state at $x = 0$, we obtain

$$\mathrm{d}x = \left(\eta - \frac{\lambda}{2}\right)\left(1+x^2\right)^{\eta-1} x\mathrm{d}t_s + \left(1+x^2\right)^{\frac{\eta}{2}}\mathrm{d}W_s, \tag{4}$$

here x is momentary return. Thus in order for the solutions of Eq. (4) to be compared with empirical compounded return they should be integrated and normalized in relevant time intervals,

$$X_{\tau_s}(t_s) = \frac{\bar{r}_0}{\tau_s}\left|\int_{t_s}^{t_s+\tau_s} x(s)\mathrm{d}s\right| \tag{5}$$

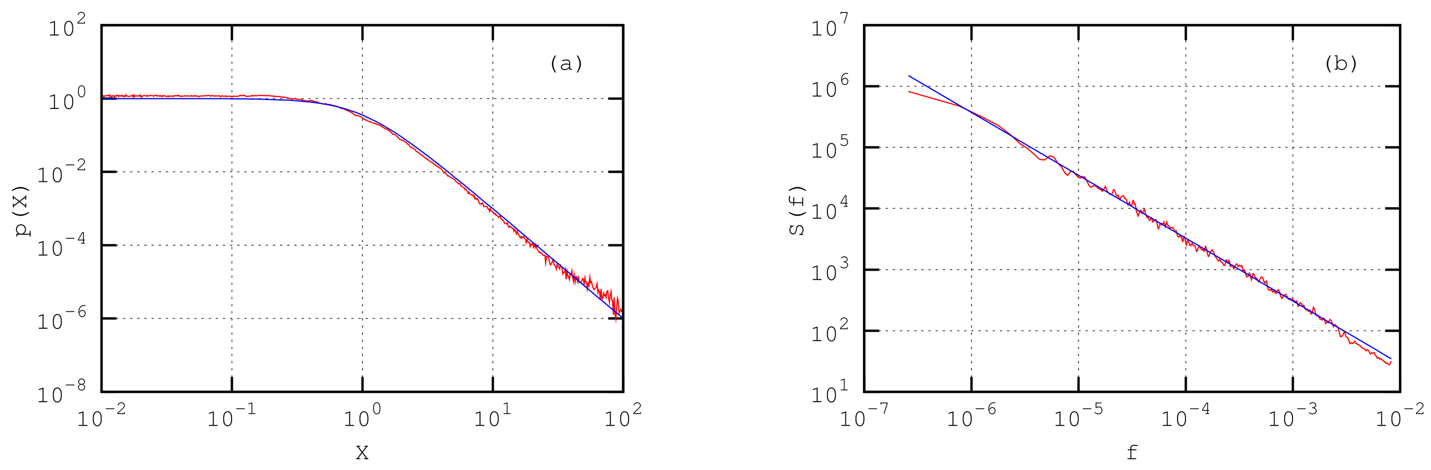

Figure 1: Statistical properties, (a) PDF and (b) PSD, obtained from time series generated by numerically solving Eq. (4) (red curves) and fitting functions (blue curves) - (a) q-Gaussian, $p(x)$, with $\lambda = 3$ and (b) power law, $1/f^\beta$, with $\beta = 1.02$. Model parameters were set as follows: $\eta = 2.5$, $\lambda = 3$, $\bar{r}_0 = 1$, $\tau_s = 2\cdot 10^{-5}$.

Sophisticated model reproducing fractured PSD

Empirical data doesn't exhibit $1/f$ noise, it actually exhibits fractured spectral density. In order to reproduce more sophisticated dynamics we need also more sophisticated SDE. Having in mind statistical features of the simple version described before and results of numerical modeling with more sophisticated versions of the SDE, we propose to model return using SDE combining two powers of multiplicativity:

$$\mathrm{d}x = \left[\eta - \frac{\lambda}{2} - \left(\frac{x}{x_{max}}\right)^2\right]\frac{(1+x^2)^{\eta-1}}{(\epsilon\sqrt{1+x^2}+1)^2}x\mathrm{d}t_s + \frac{(1+x^2)^{\frac{\eta}{2}}}{\epsilon\sqrt{1+x^2}+1}\mathrm{d}W_s, \tag{6}$$

here we have introduced parameter ϵ which divides are of x diffusion into to separate regions with differing powers of multiplicativity, and x_{max} which helps to restrict x from diverging towards infinity of either sign.

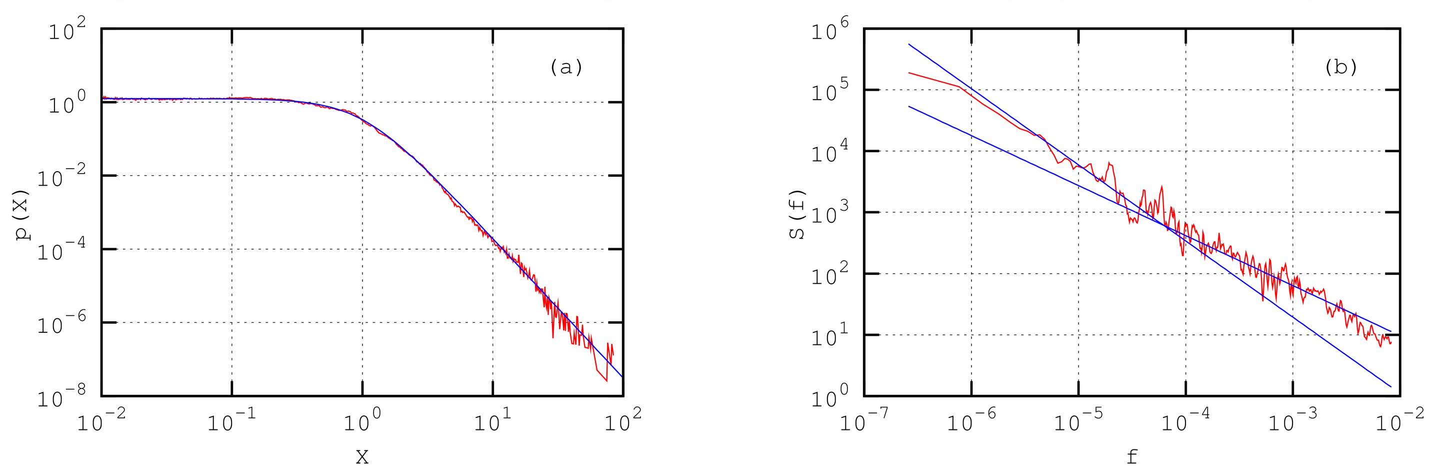

Figure 2: Statistical properties, (a) PDF and (b) PSD, obtained from time series generated by numerically solving Eq. (6) (red curves) and fitting functions (blue curves) - (a) q-Gaussian, $p(x)$, with $\lambda = 3.6$ and (b) power laws, $1/f^\beta$, with $\beta_1 = 1.24$ and $\beta_2 = 0.82$. Model parameters were set as follows: $\eta = 2.5$, $\lambda = 3.6$, $\bar{r}_0 = 1$, $\tau_s = 10^{-4}$, $\epsilon = 0.01$, $x_{max} = 10^3$.

Double stochastic model of return

Model PSD in Figure 2 (b) has overly high values of β if compared with empirical data. Though we can decrease them by assuming that there are two processes - long-range memory process described by the SDE (6) and momentary fluctuation process, which we propose to model as q-Gaussian noise with constant power law tail exponent, $\lambda_2 = 5$, and modulated variance related parameter

$$r_0(t_s, \tau_s) = 1 + X_{\tau_s}(t_s). \tag{7}$$

We can justify introduction of secondary stochastic process and from empirical point of view - in [6] we have shown that one minute return moving average of one hour correlates with fluctuating trading activity. This fact suggests us that in actual markets return relates with both trading activity, which is confirmed to have long-range memory, within the market and momentary moods.

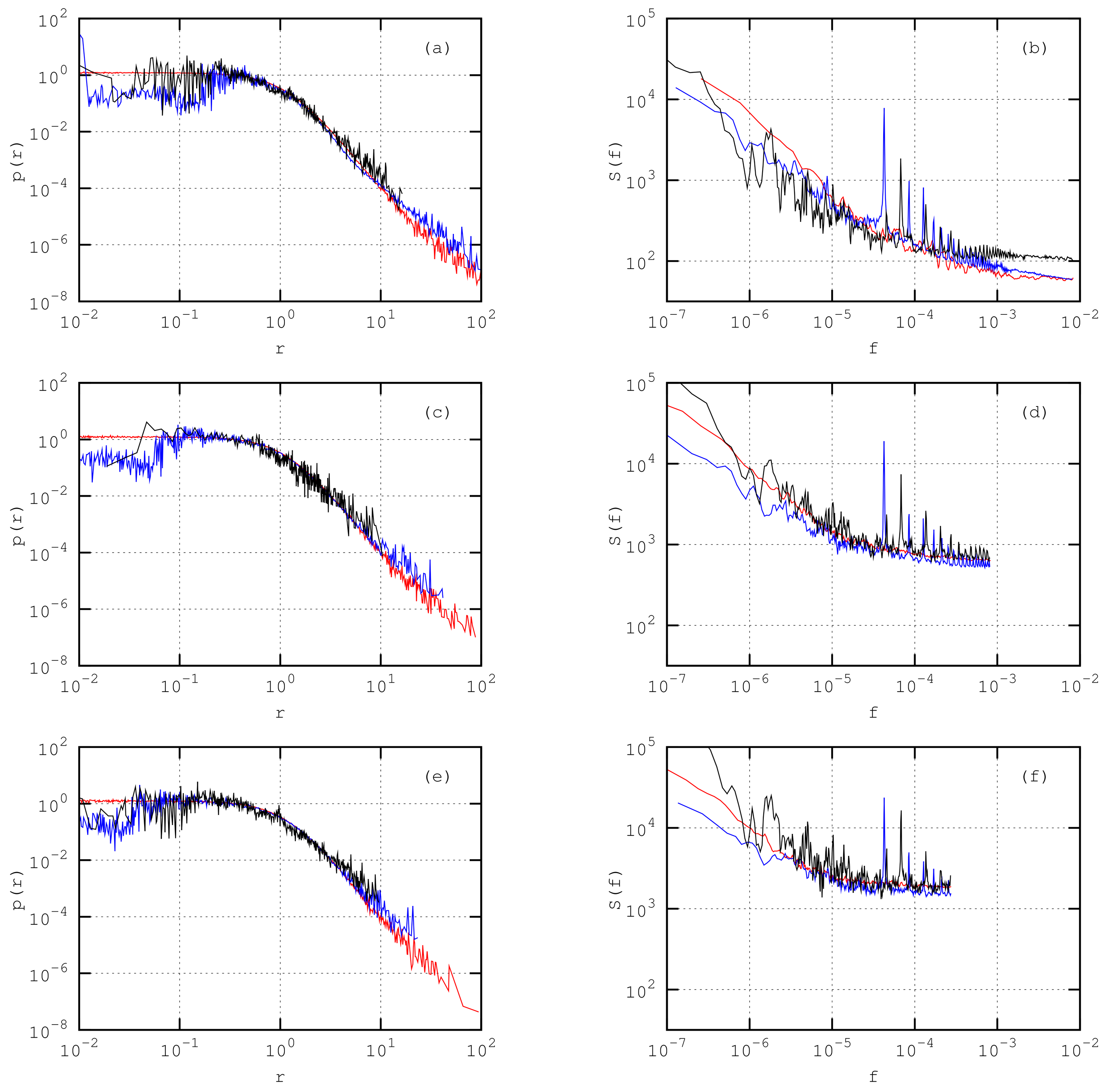

Figure 3: Agreement of model (red curves) and empirical (NYSE - blue curves, VSE - black curves) statistical properties, (a), (c) and (e) PDF and (b), (d) and (f) PSD, at different time scales - (a) and (b) 1 minute, (c) and (d) 10 minute, (e) and (f) 30 minute returns. Model parameters were set as follows: $\eta = 2.5$, $\lambda = 3.6$, $\lambda_2 = 5$, $\bar{r}_0 = 0.4$, $\tau = 2\cdot 10^{-5}/\sigma^2 = 60\,\mathrm{s}$, $\epsilon = 0.017$, $x_{max} = 10^3$.

Conclusion

In Figure 3 we show that proposed model is in excellent agreement with empirical data, which was drawn from two marginally different markets - high liquidity New York Stock Exchange (abbr. NYSE; 24 stocks traded for 27 months since January, 2005) and low liquidity NASDAQ OMX Vilnius Stock Exchange (abbr. VSE; 4 stocks traded for 50 months since May, 2005). Having achieved success in macroscopic stochastic modeling we are currently working on microscopical explanation of mechanics behind our model. Research based around agent based models, which are able to reproduce stylized facts, is ongoing.

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