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ON NETWORK THEORY

by

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SUMMARY

This paper presents a rigorous mathematical treatment of the theory of networks, culminating in a proof of the Fulkerson-Ford Min-Cut Theorem. It makes clear the intimate connection between the theory of connected graphs, on which much literature is available, and the theory of flows through a network.

ON NETWORK THEORY

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1. Introduction

Recently T. E. Harris has been concerned with the flow of trains through a network of rail lines and junction points of the rail lines. As a consequence much discussion arose in the linear programming seminar about the general problem of flow through a network. At the beginning these discussions were noticeably hampered by the lack of a precise terminology and a unified theory. Alex Boldyreff gave a number of suggestions and references in this connection and gradually a terminology and indeed a mathematical theory evolved. This work culminated in the proof of the min cut max flow theorem first conjectured by D. R. Fulkerson.

It is not to be pretended that the theory presented here is new (see for instance [1]) but the interest in problems concerning networks and flows through them seems to justify a paper presenting a unified mathematical treatment. It is with this in mind that this paper has been written.

2. Graphs and Networks

A graph G is a finite collection π of points $\{P_i\}$ (i=0,1,2,...,n) and a subset A of the product set $\pi \times \pi$ (i.e., elements of the form (P_i,P_j)). The element (P_i,P_j) has a natural geometric representation as an arc a_{ij} from P_i to P_j passing through no other point of π. For this reason elements of A will be called arcs. The points of π will be called nodes. A graph consisting of two nodes and a single arc joining them will be called a link.

A chain in a graph G from a node P_i to a node P_j is a sequence of links in $G\left\{P_iP_{j_1}, P_{j_1}P_{j_2}, P_{j_2}P_{j_3}, \ldots, P_{j_k}P_j\right\}$, where a pair $P_\ell P_m$ denotes the link consisting of $P_\ell \in \pi$, $P_m \in \pi$ and the arc $a_{\ell m}$ joining P_ℓ and P_m. The points P_i and P_j are called endpoints of the chain.

A pair of nodes in G will be said to be connected if there is a chain in G which joins them. A graph G will be called a connected graph if every pair of nodes in G is connected. A pair of nodes are said to be neighbors if there is a link in G which contains both of them. The neighborhood N_i of a node P_i is the set of all the neighbors of P_i.

In a graph G we choose a node which we call the origin and a different node P_n which we call the terminal (without loss of generality we henceforth let P_o and P_n be the origin and terminal, respectively). In addition we assign to each arc a_{ij} of G a positive number $c_{ij} = c_{ji}$ which we call the capacity of that arc. This weighted graph with an origin and a terminal is called a network. In a network, unless specifically stated otherwise, by a chain we shall mean a chain from P_o to P_n.

By a flow through a network we mean a set of real numbers $\{x_{ij}\}$, with x_{ij} assigned to the arc a_{ij}, satisfying the restrictions:

(1) $x_{oj} \geq 0$ and $x_{in} \geq 0$ (all $j \ni P_j \in N_o$ and all $i \ni P_i \in N_n$)

where we interpret the symbols to mean all (i,j) such that P_j belongs to N_o and P_i belongs to N_n,

(2) $$|x_{ij}| \leq c_{ij}$$

(3) $$\sum_{i \ni P_i \in N_j} x_{ij} = 0 \qquad (j=1,2,\ldots,n-1)$$

We further adopt the rule that $x_{ij} = -x_{ji}$. The value F_{on} of the flow is defined by

(4) $$F_{on} = \sum_{j \ni P_j \in N_o} x_{oj}$$

Theorem 1: If N is a network with a flow $\{x_{ij}\}$, and F_{on} is the value of the flow, then

$$(5) \qquad F_{on} = \sum_{j \ni P_j \in N_o} x_{oj} = \sum_{i \ni P_i \in N_n} x_{in}$$

Proof: From (3) we may write

$$(6) \quad \sum_{j=0}^{n} \sum_{i \ni P_i \in N_j} x_{ij} = \sum_{j=1}^{n-1} \sum_{i \ni P_i \in N_j} x_{ij} + \sum_{i \ni P_i \in N_o} x_{io} + \sum_{i \ni P_i \in N_n} x_{in}$$

$$= \sum_{i \ni P_i \in N_n} x_{in} - \sum_{j \ni P_j \in N_o} x_{oj}$$

But we note in the double sum on the left side of (6) that each x_{ij} occurs exactly once and that x_{ji} also occurs exactly once (since $P_i \in N_j$ implies that $P_j \in N_i$), and thus by our sign rule the double sum vanishes and the theorem follows. It is of interest to note that condition (1) is not essential for the theorems contained in this paper to hold.

3. Disconnecting Sets and Cuts

If G is a graph with a node set π, and π_1 and π_2 are disjunct subsets of π, we shall say that a set Γ of nodes of G separates π_1 and π_2 if every chain beginning in π_1 and ending in π_2 contains a point of Γ. Similarly a set D of arcs in G will be said to disconnect (to distinguish it from a node set) π_1 and π_2 if every chain beginning in π_1 and ending in π_2 contains an arc of D. If N is a network and π_1 is the origin, π_2 the terminal, and D disconnects π_1 and π_2, we shall simply call D a disconnecting set. It is clear that at least one such set exists, for the set A of all arcs is a disconnecting set. A disconnecting set which contains no proper subset which is a disconnecting set will be called a cut. Clearly every disconnecting set contains a cut.

Associated with each disconnecting set D of a network N there are two (possibly overlapping) sets of nodes:

(1) the set R(D) of all nodes of N which are not connected to P_o when D is deleted from N; and

(2) the set L(D) of all nodes of N which are not connected to P_n when D is deleted from N.

Clearly R(D) and L(D) are not empty since, by the definition of a disconnecting set, P_o is in L(D) and P_n is in R(D). The set R(D) will simply be called the right set and L(D) the left set defined by D.

A chain c from P_i to P_j will be called an <u>elementary chain</u> if it contains no proper subset which is a chain from P_i to P_j. Clearly every chain contains an elementary chain. A network in which every node is contained in a chain will be called <u>proper</u>.

<u>Theorem 2</u>: If N is a proper network and D is a cut of N, then L(D) and R(D) are disjunct.

Proof: Suppose $P_k \in R(D) \cap L(D)$. Since N is proper, P_k is contained in a chain $\left\{P_oP_{i_1}, P_{i_1}P_{i_2}, \ldots, P_{i_\ell}P_k, P_kP_{i_{\ell+1}}, \ldots, P_mP_n\right\}$. But $\left\{P_oP_{i_1}, P_{i_1}P_{i_2}, \ldots, P_{i_\ell}P_k\right\}$ is a chain from P_o to P_k and since $P_k \in R(D)$ this chain contains an element of D. Let a_{ij} be the arc in this chain which is in D and such that no succeeding arc of the chain is in D (i.e., the subchain* from P_j to P_k contains no element of D). Form D' by deleting a_{ij} from D. Then D' cannot be a disconnecting set (D is a cut); hence there is an elementary chain which contains no element of D', and thus contains a_{ij} since D is a disconnecting set. But then P_j must be connected to either P_o or P_n by a chain which contains

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*By a <u>subchain</u> of a chain c we mean a subset $\overline{c}$ of links in c which is itself a chain but not necessarily having the same endpoints as c.

no element of D; this chain, together with the chain from P_j to P_k, forms a chain from either P_o or P_n to P_k which contains no element of D. This contradicts the assumption that $P_k \in R(D) \cap L(D)$. Thus we see that R(D) and L(D) are disjunct. Theorem 2 is equivalent to:

Corollary 1: A cut divides the nodes of a proper network into two disjunct and exhaustive sets, one containing the origin and the other containing the terminal.

Hereafter all networks will be assumed to be proper.

4. Value of a Cut, Flow Across a Cut

The value of a cut D is denoted by v(D) and is the sum of the capacities of those arcs which are contained in D (i.e., $v(D) = \sum_{a_{ij} \in D} c_{ij}$).

If N is a network with a flow $\{x_{ij}\}$, the flow F(D) through a cut D is the sum of the flows x_{ij} on all those arcs $a_{ij} \in D$ where $P_i \in L(D)$ and $P_j \in R(D)$. Thus we write

$$(7) \qquad F(D) = \sum_{a_{ij} \in D \ni P_i \in L(D)} x_{ij}$$

It is clear from condition (2) that $v(D) \geq F(D)$ for any flow $\{x_{ij}\}$ and any cut D.

Theorem 3: If N is a network with a flow $\{x_{ij}\}$, and D is any cut of N, we have

(8) $$F(D) = F_{on}$$

Proof: From (3) we may write

(9) $$\sum_{j \ni P_j \in L(D)} \sum_{i \ni P_i \in N_j} x_{ij} + \sum_{a_{ij} \in D \ni P_i \in L(D)} x_{ij} = F(D) - F_{on}$$

But the left side of (9) vanishes, for whenever x_{ij} occurs so also does x_{ji}; thus (8) follows.

Corollary 2: If N is a network with a flow $\{x_{ij}\}$, and S is the class of all cuts of N, we have

(10) $$F_{on} \leq \min_{D \in S} v(D)$$

Proof: Let D_o be the cut for which $v(D)$ is a minimum (hereafter called a minimum cut). Then $F_{on} = F(D_o) \leq v(D_o) = \min_{D \in S} v(D)$.

Let H be the set of values of flows through a network N. Then $\sup_{F_{on} \in H} F_{on}$ is called the <u>capacity</u> of N and is denoted by c(N). It is clear from corollary 2 that

$$(11) \qquad c(N) = \sup_{F_{on} \in H} F_{on} \leq \min_{D \in S} v(D) = v(D_o)$$

where S is the class of cuts of N.

5. Min-Cut Theorem and Menger's Theorem

The-min cut theorem due to Fulkerson and Ford [2] asserts equality in (11). In order to prove this theorem we first prove a lemma and a theorem due to Menger [1]. The proof is along the lines of Menger's original proof; we include it here since it is not otherwise available in English.

Lemma 1: Let π be the set of nodes of a finite graph G, and let π_1 and π_2 be disjunct subsets of π such that $\pi = \pi_1 + \pi_2$. If G is such that every arc of G joins a π_1 point to a π_2 point, and π_1 cannot be separated from π_2 in G by fewer than n nodes, then there are n links in G which pairwise have no common endpoints.

Proof: Let k be the maximal number of links in G which pairwise have no common endpoints, and let this set of links be:

$$K = (P_1Q_1, P_2Q_2, P_3Q_3, \dots, P_kQ_k)$$

where $P_i \in \pi_1$ and $Q_i \in \pi_2$. We shall denote the set of nodes $\{P_i\}$ by π_1' and $\{Q_i\}$ by π_2'. A chain $(A_1A_2, A_2A_3, \dots, A_{2r-1}A_{2r})$ in G $(r \geq 1)$ will be called of K-type if the second A_2A_3, fourth $A_4A_5, \dots, 2\nu$-th $A_{2\nu}A_{2\nu+1}, \dots$ and next to last $A_{2r-2}A_{2r-1}$ links are in K (clearly the other links cannot be in K for links in K are pairwise disjunct).

No K-type chain can begin in a point of $\pi_1 - \pi_1'$ and end in a point of $\pi_2 - \pi_2'$. For if w were such a chain we could form $K + w - K \cap w$ (note that w would have to be an elementary chain) which would be a set of k+1 links pairwise having no common endpoints, and this would be contrary to the maximal property of k.

Now let us construct a set M of k nodes as follows: Let $M = (R_1, R_2, \dots, R_k)$. We choose $R_i = Q_i$ $(Q_i \in \pi_2')$ if there is a K-type chain from $\pi_1 - \pi_1'$ to Q_i. If there is no such chain we choose $R_i = P_i \in \pi_1'$. Thus from each link $P_iQ_i \in K$ we choose either P_i or Q_i but not both. By examining the four ways in which a link PQ can lie in G, we shall show that M separates π_1 and π_2 in G.

Case 1: $P \in \pi_1 - \pi_1'$ and $Q \in \pi_2 - \pi_2'$. But then PQ is a link with no endpoint in common with any link of K, and K + PQ is a set of k+1 links which pairwise have no common endpoints. Thus Case 1 is impossible.

Case 2: $P \in \pi_1 - \pi_1'$ and $Q \in \pi_2'$. Then $Q = Q_i \in \pi_2'$ for some i, and PQ is a K-type chain from $\pi_1 - \pi_1'$ to Q_i; thus Q is contained in M.

Case 3: $P \in \pi_1'$ and $Q \in \pi_2 - \pi_2'$. Then some PQ_i is in K, and either P or Q_i is in M. If Q_i is in M then there is a K-type chain w from some $P_o \in \pi_1 - \pi_1'$ to Q_i, and the chain w + Q_i P + PQ is a K-type chain which begins in $\pi_1 - \pi_1'$ and ends in $\pi_2 - \pi_2'$. This is impossible. Therefore $P \in M$.

Case 4: $P \in \pi_1'$ and $Q \in \pi_2'$. If PQ is in K, it is clear that either P or Q is in M. If PQ is not in K, then there is an arc PQ_i which is in K. If P is not in M, there is a K-type chain from some $P_o \in \pi_1 - \pi_1'$ to Q_i. This chain with Q_iP and PQ added is a K-type chain from $P_o \in \pi_1 - \pi_1'$ to Q; hence $Q \in M$.

This exhausts the possibilities, and we see that M must separate π_1 and π_2 in G. However, π_1 and π_2 cannot be separated in G by fewer than n nodes, so $k \geq n$ and the lemma is proved.

<u>Theorem 4</u> **(Menger's theorem): Let π_1 and π_2 be two disjunct subsets of the set π of nodes of a finite graph G. If π_1 and π_2 cannot be separated in G by fewer than n nodes of G, then there are n chains from π_1 to π_2 in G which pairwise have no common nodes.**

Proof: We can delete from G a sequence of arcs (perhaps null) until the property that π_1 be separated from π_2 by no fewer than n nodes will be destroyed by the deletion of one more arc. This then gives us a subgraph G' of G. If the theorem is true for G' it is true for G. We may thus consider at the outset only those graphs G which have the property:

(a) If any arc of G be deleted, π_1 and π_2 can be separated in G by fewer than n nodes of G.

If the sets π_1 and π_2 are exhaustive (i.e., $\pi = \pi_1 + \pi_2$), G can contain no arc joining two π_1 or two π_2 points, for such an arc may be deleted contrary to property (a). Hence every arc of G connects a π_1 point with a π_2 point, and the conditions of the lemma are satisfied. The other possibility is that G contains a node P_0 which is in neither π_1 nor π_2. If G contains only one arc, the theorem holds; **thus** we may make use of a complete induction on the number of arcs (i.e., we shall assume the theorem true for all graphs of less than n arcs and prove that this implies the theorem for a graph containing n arcs.).

Let us delete from G all arcs which have P_0 for an endpoint. This leaves a subgraph G' in which π_1 can be separated from π_2 by $r < n$ nodes $(P_1, P_2, \ldots, P_r)$, but the set $M = (P_0, P_1, P_2, \ldots, P_r)$ is a set of $r + 1$ nodes which separates π_1 from π_2 in G, and hence $r + 1 > n$. These two inequalities thus imply $r = n - 1$, and M has exactly n elements.

Divide M into three sets of nodes M_o, M_1, M_2, where P_i is in M_1, M_2, or M_o according as it is in π_1, π_2, or neither (thus $P_o \in M_o$). Hence $M = M_o + M_1 + M_2$, where M_o, M_1, and M_2 are pairwise disjunct.

We shall say a chain is of <u>type</u> w_1 if it connects a point of $\pi_1 - M_1$ with a point of $M_o + M_2$ and passes through no other point of M. The set of all chains of type w_1 forms a subgraph G_1 of G. Similarly we define chains of <u>type</u> w_2 to be those chains which connect a point of $\pi_2 - M_2$ with a point of $M_o + M_1$ and pass through no other point of M. The set of all chains of type w_2 forms a subgraph G_2 of G. We shall show that all nodes which are common to G_1 and G_2 belong to M. Thus if a chain $(RP_{i_1},\ldots,P_{i_k}A,AP_{i_{k+1}},\ldots,P_{i_{\ell-1}}P_{i_\ell})$ is of type w_1, and $(QP_{j_1},\ldots,P_{j_s}A,AP_{j_{s+1}},\ldots,P_{i_{r-1}}P_{i_r})$ is of type w_2, then $(RP_{i_1},\ldots,P_{i_k}A)$ and $(QP_{j_1},\ldots,P_{j_s}A)$ together form a chain from π_1 to π_2 containing no point of M; this is impossible.

From the preceding remarks, G_1 and G_2 can contain no common arcs. For if an arc were common to G_1 and G_2, so would both endpoints be common, and would thus both have to belong to M; but no chain of G_1 or G_2 can contain two points of M, and thus no arc joining two points of M can belong to either G_1 or G_2.

Through P_o there is a chain from π_1 to π_2 which passes through no other point of M (otherwise π_1 would already be separated from π_2 by a set of $n - 1$ nodes). Hence this chain decomposes into two chains, one from $\pi_1 - M_1$ to P_o and the other from $\pi_2 - M_2$ to P_o. The first is of type w_1 and the second of type w_2. Hence neither G_1 nor G_2 is the null graph, and G_1 as well as G_2 contains fewer arcs than G.

Let m_o, m_1, m_2 be the number of elements in the sets M_o, M_1, M_2, respectively. In G_1, $\pi_1 - M_1$ cannot be separated from $M_o + M_2$ by fewer than $m_o + m_2$ points. For if M' separates $\pi_1 - M_1$ from $M_o + M_2$ in G_1, then $M' + M_1$ separates π_1 from π_2 in M, and $M' + M_1 \geq n = m_o + m_2 + m_1$ implies $M' \geq m_o + m_2$. To see that $M' + M_1$ separates π_1 from π_2 in G, let w be any chain from π_1 to π_2 which contains no element of M_1. Then w must begin in $\pi_1 - M_1$ and must contain an element of $M_o + M_2$; **therefore** w contains a chain of type w_1 which must contain an element of M'. Hence w contains an element of M'.

Since G_1 possesses fewer arcs than G, we may apply our inductive assumption and say there are $m_o + m_2$ pairwise disjunct chains from $\pi_1 - M_1$ to $M_o + M_2$ in G_1. Since $m_o + m_2$ is the number of elements of $M_o + M_2$, each point in $M_o + M_2$ belongs to one and only one of these chains as an endpoint, whereas a point of M_1 which does not belong to G_1 is not in any of these chains. Let the chains passing through the points of M_2 be $T_1, T_2, \ldots, T_{m_2}$, and those through the points of M_o be $U_1', U_2', \ldots, U_{m_o}'$. In like manner one defines, reversing the roles of G_1 and G_2, $m_o + m_1$ chains of G_2: $V_1, V_2, \ldots, V_{m_1}$, and $U''_1, U''_2, \ldots, U''_{m_o}$. If U'_i and U''_j pass through the same point of M_o, then together they form a chain from π_1 to π_2. There are m_o such chains from π_1 to π_2: $U_1, U_2, \ldots, U_{m_o}$. This follows from the fact that a common point of G_1 and G_2 must belong to M. On the same grounds, the $m_o + m_1 + m_2 = n$ chains $U_1, U_2, \ldots, U_{m_o}$; $V_1, V_2, \ldots, V_{m_1}$; $T_1, T_2, \ldots, T_{m_2}$ of G are pairwise disjunct.

Corollary 3: If G is a finite graph and π_1 and π_2 are disjunct subsets of the node set π, and if π_1 and π_2 cannot be separated by fewer than n nodes in G, then the maximum number of chains from π_1 to π_2 which pairwise have no common nodes is n.

Proof: There are at most n such chains since every chain must contain an element of the minimum separating set. This along with theorem 4 establishes the corollary.

6. Proof of the Min-Cut Theorem

Now with Menger's theorem as a tool, let us return to the min-cut theorem. Let us assume, for the following theorems 5 and 6, that the capacities c_{ij} in the network N are integers. We now replace each arc a_{ij} by c_{ij} parallel* arcs a_{ijk} of

*Two arcs will be said to be parallel if they join the same pair of nodes.

unit capacity to obtain a new network* N'. It is clear that $c(N') = c(N)$.

Theorem 5: Let N be a network for which the c_{ij} are integers and let N' be its associated network, formed as above. If a cut of N' contains one of the parallel arcs joining P_i' and P_j', it must contain them all.

Proof: Suppose that a_{ij1} is contained in a cut D' of N' and that a_{ij2} is not in D'. There is a chain which contains a_{ij1} and no other element of D' (otherwise a_{ij1} could be deleted from D' and a disconnecting set would still remain). But if the link containing P_i, P_j, and a_{ij1} in this chain is replaced by the link P_i, P_j, a_{ij2}, a chain is formed which contains no element of D'; but D' is a disconnecting set and the theorem follows.

This theorem establishes the fact that a cut D' of N' is associated with a cut D of N, and that $v(D') = v(D)$. That a cut D of N defines a cut D' of N' is clear. Further, $v(D')$ is the number of arcs in D' since each arc has unit capacity.

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*The new configuration strictly fits our definition of a network only when we also insert a fictitious intermediate node on each of the parallel arcs.

<u>Theorem 6</u>: The capacity $c(N) = c(N')$ is at least as great as the maximum number of chains in N' which pairwise have no arcs in common.

Proof: A flow of 1 can be attached to each arc of these chains, and a flow of 0 to other arcs of N'.

We are now in a position to prove the min-cut theorem.

<u>Theorem 7</u>: If N is a network and S is the class of cuts of N then

$$c(N) = \min_{D \in S} v(D) \tag{12}$$

Proof: (a) The c_{ij} are integers:

Form the associated network N'. Replace each link of N' by a node. Whenever two links of N' have a node of N' in common, the nodes which replace these links are joined. The set π_1 is the set of nodes which replace those links of N' which contain P_o. The set π_2 is the set of nodes which replace links containing P_n. We assume that no link contains both P_o and P_n since the extension of the theorem to this case is clear. This gives us a graph G and two disjunct subsets π_1 and π_2 of nodes of G. A minimum cut of N' maps in (1 – 1) fashion into a minimum set of nodes separating π_1 and π_2 in G.

The maximum set of chains in N' which pairwise have no arcs in common maps in (1 – 1) fashion into the maximum number of chains from π_1 to π_2 in G which pairwise have no nodes in common. If H is the maximum number of such chains in N', then corollary 3 asserts that

$$H = \min_{D \in S} v(D) \tag{13}$$

Theorem 6 asserts then that

$$c(N) \geq H = \min_{D \in S} v(D) \tag{14}$$

Now (14), together with (11), establishes the theorem for case (a).

(b) The c_{ij} are rational:

Let k be a common denominator for the c_{ij}, and consider the network $\overline{N}$ formed from N by multiplying each capacity c_{ij} of N by k. The capacities kc_{ij} of arcs in $\overline{N}$ are now integers and case (a) applies to yield

$$kc(N) = c(\overline{N}) = \min_{\overline{D \in S}} v(\overline{D}) = k \min_{D \in S} v(D) \tag{15}$$

where the first and last equalities are consequences of the linearity of c(N) and v(D), respectively.

(c) The c_{ij} are arbitrary, positive, real numbers:

Make lower and upper rational approximations to the c_{ij}. Let $\underline{v}$, $\underline{c}(\underline{N})$; $\overline{v}$, $\overline{c}(\overline{N})$ be the value of the minimum cut and capacity of the network for the lower and upper rational approximations, respectively. By the definition of $c(N)$ and case (b) we have

$$(16) \qquad \underline{v} = \underline{c}(\underline{N}) \leq c(N) \leq \overline{c}(\overline{N}) = \overline{v}$$

By the continuity of v as a function of the c_{ij} we have $\overline{v} \rightarrow v$ and $\underline{v} \rightarrow v$ as the rational approximations converge to the c_{ij}, so that the theorem follows.

G. Dantzig and D. R. Fulkerson have produced a linear programming proof of the theorem and have developed as by-products efficient techniques for computing maximal flows through networks [3]. The nature of their proof makes it clear that Menger's theorem or the Fulkerson-Ford Max Flow Min Cut theorem are in reality consequences of the famous Min Max Theorem of Game Theory (or the Duality Theorem of Linear Programming).

It is clear from the preceding that the nature of flows through networks is intimately connected with the theory of graphs on which there is an abundant literature. No attempt has been made here to include an exhaustive list of references since reference 1 contains an extensive bibliography.

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