On the Solution of the Equations Obtained from the Investigation of the Linear Distribution of Galvanic Currents*

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LET there be given a system of n wires: 1, 2, $\cdots$, n joined to one another in an arbitrary fashion. If an electromotive force is in series with each of them, the necessary number of linear equations for the determination of the currents $I_1, I_2, \cdots, I_n$ flowing through the wires is found by using the following two theorems:[1]

Theorem 1

Let the wires $k_1, k_2, \cdots$, form a closed figure,[2] let w_k denote the resistance of wire k, E_k the electromotive force in series with that wire, and let E_k be considered positive in the same direction as I_k. Then, in case $I_{k_1}, I_{k_2}, \cdots$, are all considered positive in the same direction:

$$w_{k_1}I_{k_1} + w_{k_2}I_{k_2} + \cdots = E_{k_1} + E_{k_2} + \cdots$$

Theorem 2

If the wires $\lambda_1, \lambda_2, \cdots$, meet at one point, and if I_{λ_1}, $I_{\lambda_2}, \cdots$, are all considered as positively directed toward this point, then $I_{\lambda_1} + I_{\lambda_2} + \cdots = 0$.

Under the assumption that the given system of wires does not decompose into several completely separated systems,[3] I shall now prove that the solutions of the equations for $I_1, I_2, \cdots, I_n$, obtained by application of these theorems, may be found in the following way:

Let m be the number of crossing points, *i.e.*, the number of points at which two or more wires meet, and let $\mu = n - m + 1$. Then the common denominator of all quantities I is the sum of all $w_{k_1} \cdot w_{k_2} \cdots w_{k_\mu}$ for each μ elements of $w_1, w_2, \cdots, w_n$ having the property that no closed figure remains after removal of the wires $k_1, k_2, \cdots, k_\mu$. The numerator of I_λ is the sum of all $w_{k_1} \cdot w_{k_2} \cdots w_{k_{\mu-1}}$ for each $\mu - 1$ elements of $w_1, w_2, \cdots, w_n$ having the property that *one* closed figure remains after removal of wires $k_1, k_2, \cdots, k_{\mu-1}$ and that this closed figure contains λ. Each combination is multiplied by the sum of the electromotive forces located on the closed figure. The electromotive forces are considered as positive in the same direction as I_λ is considered positive.

For the sake of an easier over-all view, I shall divide the proof I give of this theorem into separate sections.

Proof 1: Let μ be the least number of wires that must be removed from an arbitrary system so that all closed figures are destroyed. Then μ is also the number of independent equations that can be derived by the use of Theorem 1.

In the following way one can form μ independent equations from which can be derived each equation which is a consequence of Theorem 1:

Let $1, 2, \cdots, \mu - 1, \mu$ be μ wires after whose removal no closed figure is left. After removal of $\mu - 1$ of these *one* closed figure remains. Let Theorem 1 be applied to the remaining closed figure, if one removes in turn

$$\begin{array}{l} 2, 3, \cdots, \mu - 1, \mu \\ 1, \quad 3, \cdots, \mu - 1, \mu \\ 1, 2, 3, \cdots, \mu - 1. \end{array}$$

None of the μ equations formed in this manner can be a consequence of the others, because each one contains an unknown that does not occur in any other. Only the first equation contains I_1, the second I_2, and so forth. But every equation that is a consequence of Theorem 1 can be formed from these equations. An equation for a closed figure which may be joined together from several closed figures must be formed (by addition or subtraction) from the equations for those closed figures. As we wish to show, each closed figure can be joined together from those μ figures. For all closed figures of the given system, which we shall designate by S, may be divided into those in which the wire μ occurs, and into those which are contained in the system S' that is produced from S if the wire μ is removed. If we assume that all figures that belong to the second class may be pieced together from the first $\mu - 1$ of those μ figures, we then perceive it must be possible to join together each figure of the system S from these μ figures. For an arbitrary figure in which the wire μ occurs may be joined together from a definite figure in which μ occurs and from such in which μ does not occur. The assumption made about the system S' may be reduced to a similar one about S'', if S'' is the system that is generated from S by removal of μ and $\mu - 1$; that is, it may be reduced to the assumption that all closed figures occurring in S'' may be put together from the first $\mu - 2$ of those μ figures. By continuation of this method of reasoning we finally arrive at the system

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[1] *Poggendorff's Annals*, vol. 64, p. 513.
[2] Translator's note: "closed figure" has the same meaning as the present day terms "loop," "mesh," and "circuit."
[3] Translator's note: in modern terminology, the network does not consist of several separate parts.

$S^{(\mu-1)}$. Since this contains only one closed figure, the correctness of the assumption that we must make in order to see the truth of our assertion is clear.

Proof 2: Since Theorems 1 and 2 must furnish the necessary number of equations for the determination of $I_1, I_2, \cdots, I_n$, these equations will be the following according to what we have just proved:

$$\alpha_1^1 w_1 I_1 + \alpha_2^1 w_2 I_2 + \cdots + \alpha_n^1 w_n I_n = \alpha_1^1 E_1 + \alpha_2^1 E_2 + \cdots + \alpha_n^1 E_n$$
$$\alpha_1^2 w_1 I_1 + \alpha_2^2 w_2 I_2 + \cdots + \alpha_n^2 w_n I_n = \alpha_1^2 E_1 + \alpha_2^2 E_2 + \cdots + \alpha_n^2 E_n$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\alpha_1^\mu w_1 I_1 + \alpha_2^\mu w_2 I_2 + \cdots + \alpha_n^\mu w_n I_n = \alpha_1^\mu E_1 + \alpha_2^\mu E_2 + \cdots + \alpha_n^\mu E_n$$
$$\alpha_1^{\mu+1} I_1 + \alpha_2^{\mu+1} I_2 + \cdots + \alpha_n^{\mu+1} I_n = 0$$
$$\alpha_1^{\mu+2} I_1 + \alpha_2^{\mu+2} I_2 + \cdots + \alpha_n^{\mu+2} I_n = 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\alpha_1^n I_1 + \alpha_2^n I_2 + \cdots + \alpha_n^n I_n = 0$$

where the quantities α are sometimes $+1$, sometimes -1, sometimes 0, and where μ has the same meaning as before.

It follows from this that the common denominator of the quantities I, *i.e.*, the determinant of these equations, is a homogeneous function of the μth degree of $w_1, w_2, \cdots, w_n$, which contains each w only linearly and aside from the w's contains only numbers. We can also express this result in the following way: the common denominator of the I's is the sum of each combination of μ elements of $w_1, w_2, \cdots, w_n$, where each combination is multiplied by a numerical coefficient. Likewise one notices that the numerator of each I is the sum of every combination of $\mu - 1$ elements of $w_1, w_2, \cdots, w_n$, where each combination is multiplied by a linear homogeneous function of the quantities $E_1, E_2, \cdots, E_n$ whose coefficients are numbers.

Proof 3: The observation that it is immaterial whether we make the resistance $w_\kappa = \infty$, cut the wire κ, or remove it leads to the determination of the numerical coefficients of the denominators and numerators of the quantities I. Therefore the expressions for the I's must transform, by the substitution $w_\kappa = \infty$, into the solutions of those equations that we obtain by applying Theorems 1 and 2 to the system of wires produced from the given system if we remove wire κ. I_κ itself must vanish for $w_\kappa = \infty$.

We shall divide the numerators and denominators of the I's by $w_1 \cdot w_2 \cdots w_{\mu-1}$, and then set $w_1 = \infty$, $w_2 = \infty, \cdots, w_{\mu-1} = \infty$. Let I_λ thereby transform into (I_λ). Let us designate by $A^\lambda_{\kappa_1,\kappa_2,\ldots,\kappa_{\mu-1}}$ the function of the E's which is multiplied by $w_{\kappa_1} \cdot w_{\kappa_2} \cdots w_{\kappa_{\mu-1}}$ in the numerator of I_λ. Let us symbolize the coefficient of $w_{\kappa_1} \cdot w_{\kappa_2} \cdots w_{\kappa_\mu}$ in the denominator by $a_{\kappa_1,\kappa_2,\ldots,\kappa_\mu}$. Then we have:

$$(I_\lambda) = \frac{A^\lambda_{1,2,\cdots,\mu-1}}{a_{1,2,\cdots,\mu-1,\mu} \cdot w_\mu + a_{1,2,\cdots,\mu-1,\mu+1} \cdot w_{\mu+1} + \cdots + a_{1,2,\cdots,\mu-1,n} w_n}.$$

It is a consequence of the previous observation that if λ occurs among $1, 2, \cdots, \mu - 1$:

$$(I_\lambda) = 0,$$

and if λ does not occur among $1, 2, \cdots, \mu - 1$:

$$(I_\lambda) = I'_\lambda,$$

where I'_λ denotes the current that flows through wire λ if wires $1, 2, \cdots, \mu - 1$ are removed.

We imagine the equations set up that are yielded by by applying Theorems 1 and 2 to the remaining system of wires for the determination of $I'_\mu, I'_{\mu+1}, \cdots, I'_n$. Let Theorem 1 furnish μ' independent equations. Then the common denominator of the quantities I' is a function of the (μ')th degree of $w_\mu, w_{\mu+1}, \cdots, w_n$, and the numerators are functions of the $(\mu' - 1)$th degree in the same arguments. Because of the definition of μ, either $\mu' = 1$ or $\mu' > 1$. If $\mu' > 1$, in order that the equation $(I_\lambda) = I'_\lambda$ can hold, either the numerator and denominator of I'_λ have a common factor of the $(\mu' - 1)$th degree in $w_\mu, w_{\mu+1}, \cdots$, or $(I_\lambda) = I'_\lambda = 0$, or finally (I_λ) must assume the form 0/0. If one of the quantities (I) appears in the form 0/0, then all of them must appear in the same form since they have a common denominator and none can become infinite. Should this case not occur, then numerator and denominator of each I' must have a common factor of the $(\mu' - 1)$th degree; moreover these factors must be the same for all quantities I'. However this is impossible, as can be shown in the following way.

We assume that there is a factor of the designated type which contains the quantity w_κ. κ must then be a wire that lies in at least one closed figure because otherwise w_κ could certainly not occur in the equations for I_μ, $I_{\mu+1}, \cdots$. Since the numerators and the denominators of the quantities I' are linear in each w, by removal of that factor we obtain expressions for the I's which are free of w_κ. If we substitute them into one of the equations which contains $w_\kappa I'_\kappa$, then this becomes an identity. By partial differentiation with respect to w_κ, we obtain:

$$I'_\kappa = 0.$$

But this equation cannot possibly always hold. Should it always hold, then it would have to remain correct if one sets arbitrarily many of the quantities $w = \infty$; that is, if one removes arbitrarily many of the wires. But if so many wires are removed that only *one* closed figure containing κ is left, then it is not possible that I'_κ vanish for arbitrary values of the quantities E.

We realize according to this that if $\mu' > 1$, (I_μ), $(I_{\mu+1}), \cdots, (I_n)$ must appear in the form 0/0; or since we have found $(I_1) = 0, (I_2) = 0, \cdots, (I_{\mu-1}) = 0$, if more than one closed figure remains after removal of the wires $1, 2, \cdots, \mu - 1$, the product $w_1 \cdot w_2 \cdots w_{\kappa-1}$ can occur neither in a numerator nor in the denominator of the quantities $I_1, I_2, \cdots, I_n$.

Proof 4: Now we want to try to determine the factors by which the product $w_1 \cdot w_2 \cdots w_{\mu-1}$ is multiplied in the numerators and the denominators of the I's if the condition is fulfilled that only one closed figure remains after removal of $1, 2, \cdots, \mu - 1$.

Let the figure that is left contain wires: $\lambda_1, \lambda_2, \cdots, \lambda_\nu$. Then if λ does not occur among these, $I'_\lambda = 0$, and if λ does occur among them:

$$I'_\lambda = \frac{E_{\lambda_1} + E_{\lambda_2} + \cdots + E_{\lambda_\nu}}{w_{\lambda_1} + w_{\lambda_2} + \cdots + w_{\lambda_\nu}}$$

where $E_{\lambda_1}, E_{\lambda_2}, \cdots$, are considered as positive in the same direction in which I_λ is considered as positive.

The denominator of this value can differ from the denominator of the quantity (I_λ), *i.e.*, from the expression:

$$a_{1,2,\cdots,\mu-1,\mu} w_\mu + a_{1,2,\cdots,\mu-1,\mu+1} w_{\mu+1} + \cdots a_{1,2,\cdots,\mu-1,n} w_n$$

only by a numerical factor. Therefore of the quantities

$$a_{1,2,\cdots,\mu-1,\mu}, \qquad a_{1,2,\cdots,\mu-1,\mu+1}, \cdots$$

all must vanish except for:

$$a_{1,2,\cdots,\mu-1,\lambda_1}, \qquad a_{1,2,\cdots,\mu-1,\lambda_2}, \cdots a_{1,2,\cdots,\mu-1,\lambda_\nu}$$

and these must be equal to one another. We infer from this that the coefficient of the combination $w_{\kappa_1} \cdot w_{\kappa_2} \cdots w_{\kappa_\mu}$ in the denominator of the quantities I can be different from 0 only if all closed figures are destroyed by removal of the wires $\kappa_1, \kappa_2, \cdots, \kappa_\mu$. We further conclude that all combinations which fulfill this condition and which contain $\mu - 1$ common factors w must have the same coefficient.

With the aid of this, it may be proved that any two combinations

$$w_{\kappa_1} \cdot w_{\kappa_2} \cdots w_{\kappa_\mu} \quad \text{and} \quad w'_{\kappa_1} \cdot w'_{\kappa_2} \cdots w'_{\kappa_\mu}$$

in the denominator of the I's must have the same coefficient, if all closed figures are destroyed by removal of the wires $\kappa_1, \kappa_2, \cdots, \kappa_\mu$ as well as by removal of the wires $\kappa'_1, \kappa'_2, \cdots, \kappa'_\mu$.

In order to carry out this proof, we make the following observations:

If all closed figures can be destroyed by removal of the wires $\kappa_1, \kappa_2, \cdots, \kappa_\mu$, then each of these wires must occur in at least one closed figure.

On the other hand, at least one of those wires must occur in every closed figure. Therefore, if we know of the wire κ' that it lies in a closed figure, then it must lie in the same closed figure as at least one of the wires $\kappa_1, \kappa_2, \cdots, \kappa_\mu$.

Furthermore, each of the wires $\kappa_1, \kappa_2, \cdots, \kappa_\mu$ must occur in a closed figure in which the $\mu - 1$ other wires do not occur: κ_μ, for example, in that which remains after removal of $\kappa_1, \kappa_2, \cdots, \kappa_{\mu-1}$ and which we shall symbolize by f_{κ_μ}. If the wire κ'_μ also lies in f_{κ_μ}, then all closed figures are also destroyed by removal of $\kappa_1, \kappa_2, \cdots, \kappa_{\mu-1}, \kappa'_\mu$. With the help of this observation it is easily seen that if we choose any closed figure f, $\mu - 1$ wires can always be found after whose removal f remains as the only closed figure. For if of the wires $\kappa_1, \kappa_2, \cdots, \kappa_\mu$, say $\kappa_1, \kappa_2, \kappa_3$ occur in f and if κ'_2 is a wire that occurs in f_{κ_2} but not in f, and κ'_3 a wire that appears in f_{κ_3} but not in f then $\kappa'_2, \kappa'_3, \kappa_4, \cdots, \kappa_\mu$ are wires of the desired kind.

We shall carry out the proof in such a way that we assume the coefficients of two combinations of the designated kind are equal to one another if they have ν common factors w, and we shall show that the coefficients of two combinations which have only $\nu - 1$ common factors must also be equal to one another. If we are successful in this, then we shall have shown the truth of the contention.

The method of proof remains the same no matter what value we determine for ν; hence we shall go through the proof only for one value of ν, for $\nu = 3$. We therefore want to demonstate that both the combinations:

$$w_{\kappa_1} \cdot w_{\kappa_2} \cdot w_{\kappa_3} \cdots w_{\kappa_\mu} \quad \text{and} \quad w_{\kappa_1} \cdot w_{\kappa_2} \cdot w_{\kappa_3'} \cdots w_{\kappa_\mu'}$$

must have the same coefficients.

In the system of wires that results from the given system if κ_1 and κ_2 are removed, all closed figures cannot be destroyed by the removal of fewer than $\mu - 2$ wires; they are destroyed by the removal of $\kappa_3, \kappa_4, \cdots, \kappa_\mu$, and by the removal of $\kappa'_3, \kappa'_4, \cdots, \kappa'_\mu$. Whence it follows κ'_3 lies in the same figure as at least one of the wires $\kappa_3, \kappa_4, \cdots, \kappa_\mu$, say κ_3. Let this be the only remaining one if $\kappa''_4, \kappa''_5, \cdots, \kappa''_\mu$ are removed. Then this same one is the only one left of the original system, if $\kappa_1, \kappa_2, \kappa''_4, \kappa''_5, \cdots, \kappa''_\mu$ are removed. Whence it follows that both combinations:

$$w_{\kappa_1} \cdot w_{\kappa_2} \cdot w_{\kappa_3} \cdot w_{\kappa_4''} \cdot w_{\kappa_5''} \cdots w_{\kappa_\mu''}$$

and

$$w_{\kappa_1} \cdot w_{\kappa_2} \cdot w_{\kappa_3'} \cdot w_{\kappa_4''} \cdot w_{\kappa_5''} \cdots w_{\kappa_\mu''}$$

which have $\mu - 1$ common factors w, must have the same coefficient. As a consequence of our assumption, however, also the combinations:

$$w_{\kappa_1} \cdot w_{\kappa_2} \cdot w_{\kappa_3} \cdot w_{\kappa_4} \cdots w_{\kappa_\mu}$$

and

$$w_{\kappa_1} \cdot w_{\kappa_2} \cdot w_{\kappa_3} \cdot w_{\kappa_4''} \cdots w_{\kappa_\mu''}$$

$$w_{\kappa_1} \cdot w_{\kappa_2} \cdot w_{\kappa_3'} \cdot w_{\kappa_4'} \cdots w_{\kappa_\mu'}$$

and

$$w_{\kappa_1} \cdot w_{\kappa_2} \cdot w_{\kappa_3'} \cdot w_{\kappa_4''} \cdots w_{\kappa_\mu''}$$

have pairwise the same coefficients. Therefore the coefficients of

$$w_{\kappa_1} \cdot w_{\kappa_2} \cdot w_{\kappa_3} \cdots w_{\kappa_\mu} \quad \text{and} \quad w_{\kappa_1} \cdot w_{\kappa_2} \cdot w_{\mu_3'} \cdots w_{\kappa_\mu'}$$

are also equal to one another.

We have proved herewith that the common denominator of the I's is the sum of those combinations of any μ elements $w_{\kappa_1}, w_{\kappa_2}, \cdots, w_{\kappa_\mu}$ of $w_1, w_2, \cdots, w_n$ which have the property that after removal of the wires $\kappa_1, \kappa_2, \cdots, \kappa_\mu$ no closed figure remains, where this sum is multiplied by a numerical coefficient. We can set the

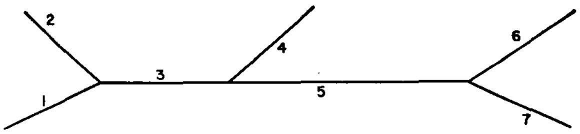

Fig. 1.

numerical coefficient equal to 1, if we adjust the numerators of the I's accordingly.

These numerators may now be found very easily. From the equations $(I_\lambda) = 0$, if $\lambda \leq \mu - 1$, and $(I_\lambda) = I'_\lambda$, if $\lambda > \mu - 1$, it follows that:

$$A^{\lambda}_{1,2,\cdots,\mu-1} = E_{\lambda_1} + E_{\lambda_2} + \cdots + E_{\lambda_\nu}$$

in case λ occurs among $\lambda_1, \lambda_2, \cdots, \lambda_\nu$, and

$$A^{\lambda}_{1,2,\cdots,\mu-1} = 0$$

in the contrary case.

The coefficient of the term $w_1 \cdot w_2 \cdots w_{\mu-1}$ equals 0 if λ does not occur in this figure. We have already shown that it can be different from 0 only if one closed figure is left after removing $1, 2, \cdots, \mu - 1$. If λ appears in the figure, then the coefficient is equal to the sum of the electromotive forces that are on the same figure, where these are considered positive in the direction in which I_λ is taken as positive.

Proof 5: Now we must still show, in order to have proved our theorem as we have formulated it, that $\mu = n - m + 1$. This contention holds only if the given system of wires does not decompose into several systems completely separated from one another, while the observations made up to now did not require such an hypothesis.

As we have seen, μ is the number of independent equations that can be derived by Theorem 1; the number of independent equations which Theorem 2 furnishes must therefore be $n - \mu$. But now it may be proved that, under that hypothesis, $n - \mu = m + 1$. It follows that $\mu = n - m + 1$.

More than $m - 1$ independent equations cannot be derived by Theorem 2. For if we apply Theorem 2 to all m crossing points, each I occurs two times in the equations thereby formed, one time with coefficient $+1$, the other time with the coefficient -1. Therefore, the sum of all equations yields the identical equation $0 = 0$. The equations obtained by application of that theorem to $m - 1$ arbitrary crossing points are, on the other hand, independent. For if we choose arbitrarily many of them, one or several of the unknowns occur only once. Let us call the crossing points $1, 2, \cdots, m$, and let (κ, λ) denote a wire through which two of them, κ and λ, are joined with one another. Then the unknown $I_{(\kappa,\lambda)}$ occurs only once in the equations for the points $\kappa_1, \kappa_2, \cdots, \kappa_\nu$, if one of them, say κ_1, is joined with another point λ, in addition to the points $\kappa_2, \cdots, \kappa_\nu$. But if the wires joining the points $\kappa_1, \kappa_2, \cdots, \kappa_\nu$ with one another do not form a closed system, one of the points $\kappa_1, \kappa_2, \cdots, \kappa_\nu$ must be connected with a point λ not in that set.

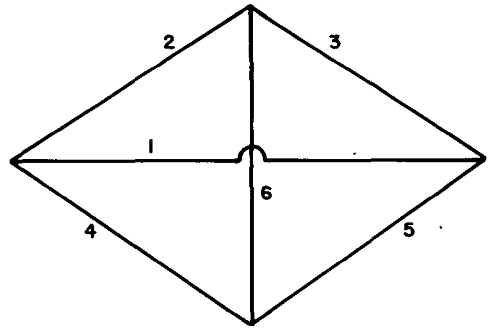

Fig. 2.

Allow me to make a few more remarks on the theorem just proved. If the terms of the numerator of I_λ are arranged in the same order as the quantities $E_1, E_2, \cdots, E_n$, then the coefficient of E_κ becomes the sum of sometimes positive, sometimes negative, combinations of any $\mu - 1$ elements of $w_1, w_2, \cdots, w_n$ which occur in the denominator of the I's multiplied by w_λ as well as by w_κ. These are exactly the combinations $w_{\kappa_1} \cdot w_{\kappa_2} \cdots w_{\kappa_{\mu-1}}$ which have the property that after removal of the wires $\kappa_1, \kappa_2, \cdots, \kappa_{\mu-1}$, only one closed figure remains and this figure contains λ as well as κ. $w_{\kappa_1} \cdot w_{\kappa_2} \cdots w_{\kappa_{\mu-1}}$ is to be taken positive if in the figure that is left the positive direction of I_λ coincides with the direction of E_κ, negative in the contrary case.

It follows from this, among other things, that if we choose two wires from an arbitrary system, the current caused in one by an electromotive force in the second is exactly equal to the current produced in the second by an equal electromotive force in the first.

The condition that we have found for the occurrence of a combination in the denominator of the I's may also be expressed, as one easily sees, in the following way: the combination $w_{\kappa_1} \cdot w_{\kappa_2} \cdots w_{\kappa_\mu}$ occurs if the equations which Theorem 1 furnishes are independent of $I_{\kappa_1}, I_{\kappa_2}, \cdots, I_{\kappa_\mu}$. It may be shown that this condition coincides with the condition that there be no equation among $I_{\kappa_1}, I_{\kappa_2}, \cdots I_{\kappa_\mu}$, or among several of these quantities, that can be derived from the equations formed by applying Theorem 2. This observation will often make it easier to represent the combinations which are missing in the denominator of the I's. If, for example, the wires 1, 2, 3 meet at a point, 3, 4, 5 at a second point, 5, 6, 7 at a third point (as in Fig. 1), all combinations are absent which contain:

$$w_1 \cdot w_2 \cdot w_3, \quad w_3 \cdot w_4 \cdot w_5, \quad w_5 \cdot w_6 \cdot w_7,$$
$$w_1 \cdot w_2 \cdot w_4 \cdot w_5, \quad w_3 \cdot w_4 \cdot w_6 \cdot w_7, \quad w_1 \cdot w_2 \cdot w_4 \cdot w_6 \cdot w_7.$$

The denominator of the I's with the combination of the wires exhibited in Fig. 2 is accordingly the sum of all combinations of any three elements of $w_1, w_2, \cdots, w_6$ with the exception of the following:

$$w_1 \cdot w_2 \cdot w_4, \quad w_1 \cdot w_3 \cdot w_5, \quad w_2 \cdot w_3 \cdot w_6, \quad w_4 \cdot w_5 \cdot w_6.$$