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Tropical Circuit Complexity

Limits of Pure Dynamic Programming

Springer

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Preface

Go to the roots of calculations! Group the operations. Classify them according to their complexities rather than their appearances! This, I believe, is the mission of future mathematicians.

–Evariste Galois

Understanding the power and weakness of algorithmic paradigms for solving decision or optimization problems in rigorous mathematical terms is an important long-term goal. Along with greedy and linear programming, dynamic programming (DP) is one of THE algorithmic paradigms for solving combinatorial optimization problems. Dynamic programming algorithms turned out to be quite powerful in many practical applications, so that we know what these algorithms *can* do. But what can DP algorithms *not* do (efficiently)? Answering this question is the subject of this book.

Roughly speaking, the idea of DP is to break up a given optimization problem into smaller subproblems in a divide-and-conquer manner and solve these subproblems recursively. Optimal solutions of smaller instances are found and retained for use in solving larger instances (smaller instances are never solved again). Many classical DP algorithms are *pure* in that they only apply the basic operations (min, $+$) or (max, $+$) in their recursion equations.

A rigorous mathematical model for pure DP algorithms is that of tropical circuits. These are conventional combinational circuits using (min, $+$) or (max, $+$) operations as gates. Pure DP algorithms are special (recursively constructed) tropical circuits. So, if one can prove that *any* tropical circuit solving a given optimization problem must use at least t gates, then we know that *no* pure DP algorithm can solve this problem by performing fewer than t (min, $+$) or (max, $+$) operations, be the designer of an algorithm even omnipotent. Thanks to the rigorous combinatorial nature of tropical circuits, ideas and arguments from the Boolean and arithmetic circuit complexity can be exploited to obtain lower bounds for topical circuits and, hence, also for pure DP algorithms.

For example, the classical Bellman–Held–Karp DP algorithm gives a tropical (min, $+$) circuit with about $n^2 2^n$ gates solving the travelling salesman problem on n-vertex graphs, while a trivial brute force algorithm results in about $n! \approx (n/\mathrm{e})^n$

gates. On the other hand, Jerrum and Snir in 1982 have shown that at least about $n^2 2^n$ gates are also necessary in any (min, +) circuit solving this problem. This shows that the Bellman–Held–Karp DP algorithm is *optimal* among all pure DP algorithms for this problem. The tropical (min, +) circuit corresponding to the (also classical) Floyd–Warshall–Roy pure DP algorithm for the all-pairs shortest paths problem on n-vertex graphs uses about n^3 gates. On the other hand, already in 1970, Kerr has shown that at least about n^3 gates are also necessary for this problem. So, the Floyd–Warshall–Roy pure DP algorithm is also optimal in the class of all pure DP algorithms.

After these and several other impressing lower bounds ~~where~~ obtained, a long break followed. Only in recent years, and mainly due to recognized connection with dynamic programming, tropical circuits have attracted growing attention again. The goal of this book is to survey the lower-bound ideas and methods that emerged during these last years.

We focus on presenting the lower-bound arguments themselves, rather than on quantitative bounds achieved using them. That is, the focus is on the proof arguments, on the ideas behind them. Because of a very pragmatic motivation of tropical circuits—their intimate relation to dynamic programming—the primary goal is to create as large as possible "toolbox" for proving lower bounds on the size of tropical circuits, not relying on unproven complexity assumptions like $\mathbf{P} \neq \mathbf{NP}$.

The difficulty in proving that a given optimization problem requires large tropical circuits lies in the nature of our adversary: the circuit. Small circuits may work in a counterintuitive fashion, using deep, devious, and fiendishly clever ideas. How can one prove that there is no clever way to quickly solve the problem? In this book, we will learn some tools to defeat this adversary.

Tropical algebra and geometry—where "adding" numbers means to take their minimum or maximum, and "multiplying" them means to add them—are now actively studied topics in mathematics. Tropical circuit complexity adds a computational complexity aspect to this topic.

The book is self-contained and is meant to be approachable already by graduate students in mathematics and computer science. The text assumes certain mathematical maturity (minor knowledge of basic concepts in graph theory, discrete probability, and linear algebra) but *no* special knowledge in the theory of computing or dynamic programming.

Supplementary material to the book can be found on my home page.

Vilnius, Lithuania/Frankfurt, Germany Stasys Jukna
June 2023

Contents

Notation

We will use more or less standard concepts and notation. For ease of reference, let us collect some of most often used ones right now:

Nonnegative real numbers	$\mathbb{R}_+ = \{x \in \mathbb{R} \colon x \geqslant 0\}$
Nonnegative integers	$\mathbb{N} = \{0, 1, 2, \ldots\}$ and $[n] = \{1, \ldots, n\}$
K_n	The complete graph on $[n]$
$K_{n,n}$	A complete bipartite $n \times n$ graph
2^X for a set X	Family of all subsets of X
$\|X\|$ for a finite set X	Number of elements in X
Family $\mathcal{F} \subseteq 2^X$ is uniform	All sets in $\mathcal{F}$ have the same cardinality
Characteristic vector of $S \subseteq [n]$	Vector $a \in \{0, 1\}^n$ with $a_i = 1$ iff $i \in S$
Unit vector $\vec{e}_i$	$\vec{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the ith position
$a \leqslant b$ for $a, b \in \mathbb{R}^n$	$a_i \leqslant b_i$ for all $i = 1, \ldots, n$
$A \subseteq \mathbb{R}^n$ is an antichain	$a \not\leqslant b$ for all $a \neq b \in A$
Upward closure $A^{\uparrow}$ of $A \subseteq \mathbb{N}^n$	$A^{\uparrow} = \{b \in \mathbb{N}^n : b \geqslant a$ for some $a \in A\}$
Downward closure $A^{\downarrow}$ of $A \subseteq \mathbb{N}^n$	$A^{\downarrow} = \{b \in \mathbb{N}^n : b \leqslant a$ for some $a \in A\}$
$B \subseteq \mathbb{R}^n$ lies above $A \subseteq \mathbb{R}^n$	$B \subseteq A^{\uparrow}$, i.e., $\forall b \in B\ \exists a \in A :\ b \geqslant a$
$B \subseteq \mathbb{R}^n$ lies below $A \subseteq \mathbb{R}^n$	$B \subseteq A^{\downarrow}$, i.e., $\forall b \in B\ \exists a \in A :\ b \leqslant a$
Support of $a \in \mathbb{R}^n$	$\sup(a) = \{i : a_i \neq 0\}$
Degree of $a \in \mathbb{N}^n$	$\|a\| = a_1 + \cdots + a_n$
Lower envelope of $A \subseteq \mathbb{N}^n$	$\lfloor A \rfloor = \{a \in A \colon \|a\|$ is minimal$\}$
Higher envelope of $A \subseteq \mathbb{N}^n$	$\lceil A \rceil = \{a \in A \colon \|a\|$ is maximal$\}$
$A \subseteq \mathbb{N}^n$ is homogeneous	$\lceil A \rceil = \lfloor A \rfloor$
Sum of $a, b \in \mathbb{R}^n$	$a + b = (a_1 + b_1, \ldots, a_n + b_n)$
Minkowski sum of $A, B \subseteq \mathbb{R}^n$	$A + B = \{a + b \colon a \in A, b \in B\}$
Scalar product of $a, b \in \mathbb{R}^n$	$\langle a, b\rangle = a_1 b_1 + \cdots + a_n b_n$
Tropical (min, +) polynomial	$f(x) = \min_{a \in A}\{\langle a, x\rangle + c_a\}$; $A \subseteq \mathbb{N}^n$, $c_a \in \mathbb{R}_+$