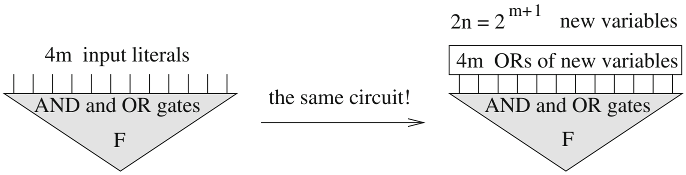

Fig. 1.14 Having a circuit F computing a boolean function f of $2m$ variables, we obtain a (monotone) circuit representing the graph G_f by replacing each input literal in F by an appropriate OR of new variables

vector with exactly one 1. In particular, parity functions also have this property, as well as any function $g(Z) = \varphi(\sum_{w \in S} z_w)$ with $\varphi : \mathbb{N} \to \{0, 1\}$, $\varphi(0) = 0$ and $\varphi(1) = 1$ does.

The Magnification Lemma is particularly appealing when dealing with circuits containing *unbounded* fanin OR (or unbounded fanin Parity gates) on the next to the input layer (Fig. 1.14). In this case the total number of gates in the circuit computing f is exactly the number of gates in the obtained circuit representing the graph G_f! Thus if we could prove that some explicit bipartite $n \times n$ graph with $n = 2^m$ cannot be represented by such a circuit of size n^ϵ, then this would immediately imply that the corresponding boolean function $f(x, y)$ in $2m$ variables cannot be computed by a (non-monotone!) circuit of size $n^\epsilon = 2^{\epsilon m}$, which is already exponential in the number of variables of f. We will use Lemma 1.32 in Sect. 11.6 to prove truly exponential lower bounds for unbounded-fanin depth-3 circuits with parity gates on the bottom layer.

It is important to note that moderate lower bounds for graphs even in very weak circuit models (where strong lower bounds for boolean functions are easy to show) would yield impressive lower bounds for boolean circuits in rather nontrivial models. To demonstrate this right now, let $\mathrm{cnf}(G)$ denote the smallest number of clauses in a monotone CNF (AND of ORs of variables) representing the graph G.

A bipartite graph is $K_{2,2}$-free if it does not have a cycle of length 4, that is, if its adjacency matrix does not have a 2×2 all-1 submatrix.

■ **Research Problem 1.33.** Does there exist a constant $\epsilon > 0$ such that $\mathrm{cnf}(G) \geq D^\epsilon$ for every bipartite $K_{2,2}$-free graph G of average degree D?

We will see later in Sect. 11.6 that a positive answer would give an explicit boolean function f of n variables such that any DeMorgan circuit of depth $\mathcal{O}(\log n)$ computing f requires $\omega(n)$ gates (cf. Research Problem 11.17). Thus graph complexity is a promising tool to prove lower bounds for boolean functions. Note, however, that even small lower bounds for graphs may be very difficult to prove. If, say, $n = 2^m$ and if $f(x, y)$ is the parity function of $2m$ variables, then any CNF for f must have at least $2^{2m-1} = n^2/2$ clauses. But the bipartite $n \times n$ graph G_f corresponding to this function consists of just two complete bipartite subgraphs; hence, G_f can be represented by a monotone CNF consisting of just four clauses.