19.10. Prove Lemma 19.40.

Hint: Argue as in the proof of Lemma 19.39 to show that there must be an i for which $\sum_{j \in I_i} a_j$ is odd. Consider the set of vectors $V \subset P$ that average to x and that differ from x exactly on I_i where they take 0-1 values. Show that $a^T v \le b$ for all $v \in V$.

19.11. The clique-coloring polytope described in Sect. 19.4 corresponds to the following maximization problem $MP_{n,l}$ for $l = k - 1$: Maximize the number of nodes in a clique of an n-vertex graph whose chromatic number does not exceed l. Although l is a trivial solution for this problem, Corollary 19.16 show that any cutting plane proof certifying that no such graph can have a clique on more than l vertices must generate an exponential number of inequalities. That is, quick cutting plane algorithms cannot solve this maximization problem optimally. Use a lower bound on the monotone circuit size of clique like functions (Theorem 9.26) to show that such algorithms cannot even *approximate* this problem: any cutting plane proof certifying that no l-colorable graph can have a clique on $k > l$ vertices must generate an exponential in $\min\{l, n/k\}^{\Omega(1)}$ number of inequalities.

19.12. ■ **Research Problem.** Given a graph $G = (V, E)$, consider the following communication game. Alice gets a subset $A \subseteq V$, Bob gets a subset $B \subseteq V$ such that $|A \cup B| > \alpha(G)$. Hence, $A \cup B$ must contain at least one edge. The goal is to find such an edge. Does there exist n-vertex graphs G for which this game requires $\omega(\log^2 n)$ bits of communication?

Comment: By Lemma 19.8, this would imply that every tree-like CP proof with bounded coefficients for the unsatisfiability of the system (19.2) augmented with the inequality $\sum_{v \in V} x_v \ge \alpha(G) + 1$ must have super-polynomial size. This would be the first strong lower bound for a non-artificial system corresponding to an important optimization problem, the maximum independent set problem.

19.13. (Split cuts) Given a polytope $P = \{x \in \mathbb{R}^n : Ax \le b\}$, a *cut* for P is any inequality $c^T x \le d$ with integral coefficients such that $c^T x \le d$ is valid in the 0-1 restriction $P \cap \{0,1\}^n$ of P. In this case one also says that $Ax \le b$ *implies* $c^T x \le d$. In CP-proofs we used simplest cuts: if $a^T x \le b$ is valid in P, and if all coordinates of a are dividable by an integer c, then $(a/c)^T x \le \lfloor b/c \rfloor$ is valid in $P \cap \{0,1\}^n$. There are also other types of cuts. An inequality $c^T x \le d$ is a *lift-and-project cut* for $P = \{x \in [0,1]^n : Ax \le b\}$ if for some index i, $c^T x \le d$ is satisfied by points in $P \cap \{x : x_i = 0\}$ and by points in $P \cap \{x : x_i = 1\}$. Even more powerful are so-called "split cuts". An inequality $c^T x \le d$ is a *split cut* for P if there exist $a \in \mathbb{Z}^n$ and $b \in \mathbb{Z}$ such that $c^T x \le d$ is satisfied in $P \cap \{x : a^T x \le b\}$ as well as in $P \cap \{x : a^T x \ge b + 1\}$; the inequality $a^T x \le b$ is a *witness* for this cut. In particular, any inequality valid in the whole polytope P is a (useless) split cut for P with the witness $0^T x \le 0$.

(a) Show that each Gomory–Chvátal cut is a special case of a split cut.
(b) Consider the polytopes $P_t = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 - 2tx_2, x_1 + 2tx_2 \le 2t\}$, $t = 1, 2, \ldots$. It was observed by J. A. Bondy that every CP-proof of $x_1 \le 0$ from $A_t x \le b_t$ using Gomory–Chvátal cuts has size at least t, which is exponential