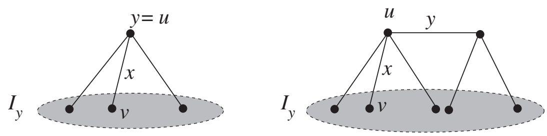


Fig. 7.1 The cases when $y \in V$ (*left*) and when $y \in E$ (*right*)

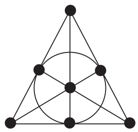
By (7.1), it remains to show that the entire matrix M has full row-rank $|E|$ over GF(2). For this, take an arbitrary subset $\emptyset \neq F \subseteq E$ of edges. We have to show that the columns of the submatrix M' of M corresponding to the rows labeled by edges in F cannot sum up to the all-0 column over GF(2).

If F is not an even factor, that is, if the number of edges in F containing some vertex v is odd, then the column of v in M' has an odd number of 1s, and we are done.

So, we may assume that F *is* an even factor. Take an arbitrary edge $y = uv \in F$, and let $H \subseteq F$ be the set of edges in F incident to at least one endpoint of y. Since both vertices u and v have even degree (in F), the edge y has a nonempty intersection with an *odd* number of edges in F: one intersection with itself and an even number of intersections with the edges in $H \setminus \{y\}$. Thus, the y-th column of M' contains an odd number of 1s, as desired. □

Explicit constructions of dense triangle-free graphs without four-cycles are known.

Example 7.4. (Point-line incidence graph) For a prime power q, a projective plane $PG(2,q)$ has $n = q^2 + q + 1$ points and n subsets of points (called lines). Every point lies in $q + 1$ lines, every line has $q + 1$ points, any two points lie on a unique line, and any two lines meet is the unique point. Here is a $PG(2,2)$, known as the Fano plane (with 7 lines and 3 points on a line):



Now, if we put points on the left side and lines on the right, and joint a point x with a line L by an edge if and only if $x \in L$, then the resulting bipartite $n \times n$ graph will have $(q + 1)n = \Theta(n^{3/2})$ edges and contain no four-cycles. The graph clearly has no triangles, since it is bipartite.

Example 7.5. (Sum-product graph) Let p be a prime number and take a bipartite $n \times n$ graph with vertices in both its parts being pairs (a,b) of elements of a finite field $\mathbb{Z}_p$; hence, $n = p^2$. We define a graph G on these vertices, where (a,b) and (c,d) are joined by an edge if and only if $ac = b + d$ (all operations modulo p). For each vertex (a,b), its neighbors are all pairs $(x, ax - b)$ with $x \in \mathbb{Z}_p$. Thus, the graph is p-regular, and has $n = np = p^3 = n^{3/2}$ edges. Finally, the graph cannot have four-cycles, because every system of two equations $ax = b + y$ and $cx = d + y$ has at most one solution.