## ON RECTIFIER AND SWITCHING-AND-RECTIFIER CIRCUITS*

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One of the problems in the theory of circuits is a synthesis of circuits implementing some functions and including possibly less elements. In many cases construction of different "devices" implies a synthesis of circuits of a common type. Let us give some examples.

1. Electric circuits, built of rectifiers (i.e. bipolar elements of one-way current conductivity) are considered. They have poles of two sorts: inputs and outputs. So the problem is to synthesize a circuit implementing given matrix of full conductivities (from inputs to outputs) via less possible number of rectifiers.

2. Assume numbers $A_1, \dots, A_p$ are given. It is required to form a sums of some of them $S_i = \sum_{k=1}^{q_i} A_{j_k}$ ($j_k \neq j_l$ if $k \neq l$) via less possible number of additions (storage of intermediate results is permitted).

These cybernetics problems, as well as some others (see e.g. [1], [2, p. 104]), lead to the notion of rectifier circuit.

1°. R e c t i f i e r is a bipolar oriented element, i.e. an element with an input pole $a$ and output pole $b$ (notation $\overrightarrow{ab}$). We will consider circuits [3] built of rectifiers (r e c t i f i e r c i r c u i t s). Term non-self-intersecting path $\overrightarrow{c_0c_1}, \overrightarrow{c_0c_1}, \dots, \overrightarrow{c_{n-1}c_n}$ as o r i e n t e d c h a i n, $c_0$ — the b e g i n n i n g, $c_n$ — the e n d of the chain.

Correspond to each (ordered) pair of poles $c, d$ of a rectifier circuit a number $(c,\ d)$, which is 1 if $c = d$ or there exists an oriented chain with the beginning $c$ and the end $d$, and which is 0 if the opposite holds.

2°. Consider one important type of rectifier circuits.

D e f i n i t i o n. R e c t i f i e r $(p,\ q)$-c i r c u i t (or, to be shorter, $(p,\ q)$-circuit) is a rectifier circuit with $p + q$ poles $a_1, \dots, a_p, b_1, \dots, b_q$ satisfying the following conditions:

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1) $(a_{i_1},\, a_{i_2}) = 0$; $i_1 \neq i_2$, $1 \leq i_1,\, i_2 \leq p$;

2) $(b_{j_1},\, b_{j_2}) = 0$; $j_1 \neq j_2$, $1 \leq j_1,\, j_2 \leq q$;

3) $(b_j,\, a_i) = 0$; $1 \leq i \leq p$, $1 \leq j \leq q$.

Poles $a_1, \ldots, a_p$ are considered as i n p u t s, poles $b_1, \ldots, b_q$ are considered as o u t p u t s of the $(p,\, q)$-circuit. Define r a n k of a $(p,\, q)$-circuit as the length of the longest oriented chain from input to output.

Assign to each $(p,\, q)$-circuit $S$ a matrix $A = ||\alpha_{ij}||$; $1 \leq i \leq p$, $1 \leq j \leq q$, $\alpha_{ij} = (a_i,\, b_j)$. Then we will define that $(p,\, q)$-circuit $S$ implements a matrix $A$. Evidently, any matrix with $p$ rows and $q$ columns (elements 0 and 1) can be implemented by some $(p,\, q)$-circuit.

Introduce the following functions: $B_r(A)$ — minimal number of rectifiers in a rectifier circuit of the rank not larger than $r$ implementing matrix $A$; $B_r(p,q) = \max B_r(A)$ (maximum covers all matrices with $p$ rows and $q$ columns); $B(A)$ — minimal number of rectifiers in a rectifier circuit implementing matrix $A$ (rank is not constrained); $B(p,q)$ is defined analogously[1].

One can easily notice that $B_r(A) = B_r(A')$ ($A'$ is a matrix transposed to $A$), and consequently $B_r(p,q) = B_r(q,p)$, $B(p,q) = B(q,p)$. Furthermore, $B(p,q) \leq B_r(p,q)$. Yet, it is evident that $B_1(A)$ is a number of ones in matrix $A$ and $B_1(p,q) = pq$.

L e m m a.

$$B_2(p,q) \leq p + q \cdot 2^{q-1}.$$

P r o o f. Divide rows of a matrix (with $p$ rows and $q$ columns) into groups of equal rows. Evidently, number $t$ of such groups does not exceed $2^q$. Assume $k$-th group contains $p_k$ rows with numbers $i_1, i_2, \ldots, i_{p_k}$. If each of these rows has $q_k$ ones (in positions $j_1, j_2, \ldots, j_{q_k}$), then the group of rows can be implemented by a $(p_k,\, q_k)$-circuit of rank 2 composed of $p_k + q_k$ rectifiers $\overrightarrow{a_{i_r} c_k}$ $(r = 1, \ldots, p_k)$, $\overrightarrow{c_k b_{j_s}}$ $(s = 1, \ldots, q_k)$, where $c_k$ is a vertex different from $a_i$ and $b_j$ (see fig. 1). Implementation of the whole matrix requires

$$\sum_{k=1}^{t} p_k + \sum_{k=1}^{t} q_k \leq p + \sum_{l=0}^{q} \sum_{q_k = l} l \leq p + \sum_{l=0}^{q} C_q^l \cdot l = p + q \cdot 2^{q-1}$$

rectifiers. Fig. 1 represents a matrix and a circuit implementing it.

[1]An analogous function was introduced for the first time by Shannon [4] for estimation the number of switches in switching circuits, implementing Boolean functions.

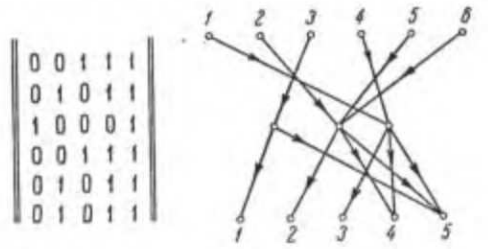


Fig. 1

T h e o r e m 1. Assume sequence $(p_1, q_1), \ldots, (p_n, q_n), \ldots$ satisfies conditions $p_n \to \infty$, $p_n \geq q_n$ and $\frac{\lg_2 p_n}{q_n} \to \infty$.

Then
$$B_2(p_n, q_n) \sim \frac{p_n q_n}{\lg_2 p_n}.$$

P r o o f. 1) U p p e r b o u n d (and a method of synthesis). A matrix to implement is divided into parts, each part contains $s_n = [\lg_2 p_n - 2\lg_2 \lg_2 p_n]$ columns (possibly one part contains less columns). Implementation of any part requires (lemma) not larger than

$$p_n + s_n \cdot 2^{s_n - 1} \leq p_n + \frac{p_n(\lg_2 p_n - 2\lg_2 \lg_2 p_n)}{2(\lg_2 p_n)^2} = p_n(1 + o(1))$$

rectifiers. The number of parts is not larger than

$$\left[\frac{q_n}{s_n}\right] + 1 \leq \frac{q_n}{s_n} + 1 < \frac{q_n}{\lg_2 p_n - 2\lg_2 \lg_2 p_n - 1} + 1 = \frac{q_n}{\lg_2 p_n}(1 + o(1)).$$

Totally not larger than $\frac{p_n q_n}{\lg_2 p_n}(1 + o(1))$ rectifiers are used.

2) L o w e r b o u n d, asymptotically equal to the upper one, follows from the fact that the number of minimal $(p, q)$-circuits containing not larger than $k$ rectifiers does not exceed $C_1^k C_2^{p+q}(p+q)^k$.

R e m a r k s. Under the conditions of the theorem 1:

1) $\frac{p_n q_n}{\lg_2 p_n + \lg_2 q_n}(1 + o(1)) < B(p_n, q_n) < \frac{p_n q_n}{\lg_2 p_n}(1 + o(1))$.

The first inequality follows from the theorem 2 [3], the second — from the theorem 1 (this paper).

2) If, in addition $\frac{\lg_2 q_n}{\lg_2 p_n} \to 0$ holds, then

$$B(p_n, q_n) \sim B_2(p_n, q_n) \sim \frac{p_n q_n}{\lg_2 p_n}.$$

3°. Let us point to one application of rectifier circuits. We will consider circuits constructed of contacts of entrance relays (switches) and rectifiers (i.e. bipolar devices with one-way current conductivity) implementing Boolean functions as c o n d u c t i v i t y f u n c t i o n s[2].

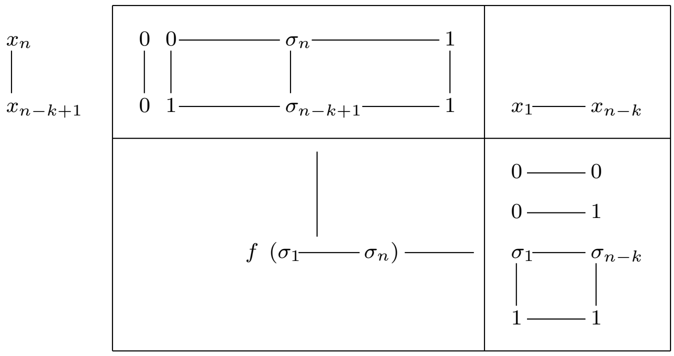
Let $L_{SR}(n)$ be the minimal total number of switches and rectifiers, allowing circuit to implement any Boolean function of $n$ variables.

T h e o r e m 2.

$$L_{SR}(n) \sim \frac{2^n}{n}.$$

P r o o f. 1) U p p e r b o u n d. Any Boolean function of $n$ variables can be represented via table $T$ with two inputs [6] (see Table 1). A circuit is constructed (according to the Table 1) of two switching trees [4] of variables $x_1, \ldots, x_{n-k}$ and $x_{n-k+1}, \ldots, x_n$ and rectifier multipole. Number of inputs of the multipole fits the number of outputs of the first tree, number of outputs of the multipole fits the number of inputs of the second tree, matrix of the full conductivities fits the table 1 (precisely, it fits the part of the table, which contains values of the function). Poles of the multipole are to be connected with the corresponding poles of the trees (fig. 2)[3].

Table 1




[2]Circuits constructed of switches and rectifiers (and also resistors) can implement Boolean functions in another way — as voltage functions (see, e.g., [5]).

[3]This approach to the synthesis of switching-and-rectifier circuits was proposed by G. N. Povarov and S. V. Yablonskiy.

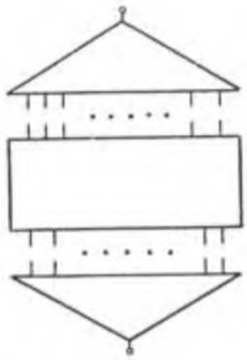


Fig. 2

Let $k = [2\lg_2 n]$. Then the number of switches in the trees is less than

$$2\left(\frac{2^n}{2^{[2\lg_2 n]}} + 2^{[2\lg_2 n]}\right) \le 2\left(\frac{2\cdot 2^n}{n^2} + n^2\right) = \frac{4\cdot 2^n}{n^2}(1+o(1)).$$

The rectifier multipole has $p_n = 2^{n-[2\lg_2 n]}$ inputs and $q_n = 2^{[2\lg_2 n]}$ outputs. It is easy to check that conditions of the theorem 1 are satisfied. Thus a rectifier $(p_n,\, q_n)$-pole circuit can be constructed, which contains not larger than $\frac{p_n q_n}{\lg_2 p_n}(1+o(1)) = \frac{2^n}{n}(1+o(1))$ rectifiers. The total number of switches and rectifiers in the circuit is also does not exceed $\frac{2^n}{n}(1+o(1))$.

2) L o w e r  b o u n d — see [3, example 1°].

4°. Define $\widetilde{L}(n)$ as the minimal number such that any Boolean function of $n$ variables can be implemented as a conductivity function by a switching-and-rectifier circuit, containing not larger than $\widetilde{L}(n)$ switches.

T h e o r e m 3. For any $\epsilon > 0$ and $n > n(\epsilon)$

$$\frac{1}{2}\cdot 2^{n/2}(1-\epsilon) < \widetilde{L}(n) < 3\sqrt{2}\cdot 2^{n/2}.$$

P r o o f. 1) U p p e r  b o u n d. Set $k = [n/2]$ in the construction from the proof of the theorem 2.

2) L o w e r  b o u n d. Evidently, if a function can be implemented by a circuit with not larger than $m$ switches it can be implemented by a circuit with exactly $m$ switches. Each such circuit can be produced from some rectifier $(2m+2)$-pole circuit via connection of switches between $(2i-1)$-th and $2i$-th poles $(i = 1, \ldots, m)$; poles of the obtained circuit are $(2m+1)$-th and $(2m+2)$-th poles of the rectifier multipole. It is easy to see that if two

rectifier multipoles has the same matrices of full conductivities, then the sets of functions, implemented by circuits obtained from any of the multipoles, are the same. Thus the number of functions implemented by circuits with not larger than $m$ switches does not exceed

$$2^{(2m+2)(2m+1)}(2n)^m = n^m \cdot 2^{4m^2+7m+2} = N_m.$$

One can verify straightforwardly that if $m \leq \frac{1}{2} \cdot 2^{n/2} - \frac{\lg_2 n+7}{8}$ then $N_m < 2^{2^n}$.

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