On the CNF-complexity of bipartite graphs containing no squares.

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Abstract

By a probabilistic construction, we find a bipartite graph having average degree d which can be expressed as a conjunctive normal form using $C \log d$ clauses. This negatively resolves Research Problem 1.33 of Jukna.

1 Introduction

We say $G = (V, W, E)$ is a bipartite graph over V and W if V and W are sets of vertices and $E \subset V \times W$ is the set of edges. Given two graphs G_1 and G_2 over V and W with $G_1 = (V, W, E_1)$ and $G_2 = (V, W, E_2)$, we may define union and intersection edge-setwise, where

$$G_1 \cup G_2 = (V, W, E_1 \cup E_2),$$

and

$$G_1 \cap G_2 = (V, W, E_1 \cap E_2).$$

We may define unions and intersections of families of bipartite graphs over V and W.

A special type of graph we consider is $CL(A, B)$, the clause graph of $A \subset V$ and $B \subset W$. Then

$$CL(A, B) = (V, W, (A \times W) \cup (V \times B)).$$

(The graph $CL(A, B)$ is called a clause graph because it is the union of all stars of vertices in A and B.)

We say that sets $A_1, \ldots, A_n \subset V$ and $B_1, \ldots, B_n \subset W$ form a conjunctive normal form using n clauses for a graph G over V and W if

$$G = \bigcap_{i=1}^{n} CL(A_i, B_i).$$

In Jukna's recent book [Juk], he poses the following conjecture as Research Problem 1.33. (In fact, the conjecture is a recurring theme in the book and reappears as Research problems 4.9 and 11.17.) A slightly stronger conjecture was made in [PRS] p. 523.

Conjecture 1.1. *There is a universal $\epsilon > 0$ so that any bipartite graph G having no $K_{2,2}$'s as subgraphs and having average degree d has no conjunctive normal form using $\lesssim d^\epsilon$ clauses.*

A positive result for Conjecture 1.1 would be important because it would allow one to construct an explicit Boolean function so that any low depth circuit computing it would require many gates. See ([Juk], Chapter 11).

Unfortunately, we prove

Theorem 1.2. *For all $\epsilon > 0$ given d sufficiently large, there is a bipartite graph G with average degree $\gtrsim d^{1-\epsilon}$ so that G has a conjunctive normal form with at most $O(\log d)$ clauses.*

has no K_{2,2}'s and

(Here we use the notation $A \gtrsim B$ to mean that there is a universal constant C, independent of d so that $CA \geq B$. We have stated theorem 1.2 in this way because d will be a parameter at the beginning of our construction. Of course $\log d \sim \log(d^{1-\epsilon})$.)

Clearly, theorem 1.2 contradicts conjecture 1.1. Indeed, we remark that aside from constants, the theorem is sharp. Given a $K_{2,2}$-free graph $G = (V, W, E)$ with average degree d, we may assume WLOG that there are at least d vertices $v_1, \ldots v_d$ of V adjacent to more than two elements of W each. We let W_v be the set of elements of W adjacent to v. Then the sets $W_{v_1}, \ldots, W_{v_d}$ are distinct since in particular each intersection of two of them contains at most one element by the $K_{2,2}$-free condition. However, if we have

$$G = \bigcap_{i=1}^{n} CL(A_i, B_i),$$

then we have

$$W_v = \bigcap_{i: v \notin A_i} B_i.$$

Thus there are at most 2^n distinct sets W_v. Hence $n \geq \log_2 d$.

We now explain the idea behind theorem 1.2. We consider the simplest model of a random bipartite graph between sets of vertices having N elements each. We choose i.i.d. Bernoulli random variables $X_{v,w}$ indexed by $V \times W$. We define the random graph

$$G = (V, W, E),$$

where

$$E = \{(v, w) : X_{v,w} = 1\}.$$

To get average degree close to d, we set the probability that a given $X_{v,w} = 1$ to be $\frac{d}{N}$. We should imagine that N is quite large compared to d, say $N = d^{10}$. We calculate the probability that there is a $K_{2,2}$ involving vertices v_1, v_2, w_1, w_2. By the independence of the random variables, clearly the probability is $\frac{d^4}{N^4}$. Thus we expect the graph G to have only d^4 copies of $K_{2,2}$. But this is quite small compared to the number of vertices of G. By removing $2d^4$ vertices, we should be able to get a $K_{2,2}$-free graph.

To prove Theorem 1.2, we will replace this simple model of a random graph by a random conjunctive normal form. We will show that it has roughly the same behavior as the random graph so that after removing a small number of vertices, which we can do without changing the number of clauses in the conjunctive normal form, we arrive at a $K_{2,2}$-free graph.

Finally, we make the remark that a simple argument using Cauchy-Schwarz shows that to get a $K_{2,2}$-free graph of average degree d on N vertices, we need $N \gtrsim d^2$. We let I_{ij} be the edge matrix of the graph ~~of the graph~~, where here i and j run from 1 to N. Then

$$\sum_{i=1}^{N} \sum_{j=1}^{N} I_{ij} = Nd.$$

On the other hand,

$$\sum_{i=1}^{N} \sum_{j=1}^{N} I_{ij} \leq N^{\frac{1}{2}} \left(\sum_{i=1}^{N} (\sum_{j=1}^{N} I_{ij})^2 \right)^{\frac{1}{2}}.$$

Now, we just expand the inside square, interchange the order of the sum and observe that

$$\sum_{i} I_{ij} I_{ik} \leq 1,$$

when $j \neq k$ by the condition that there are no $K_{2,2}$'s.

We remark that this Cauchy-Schwarz argument in fact imposes a great deal of structure on the graph G. This lends us the temerity to make the following conjecture:

Conjecture 1.3. *There is a universal $\epsilon > 0$ so that any bipartite graph G having no $K_{2,2}$'s as subgraphs and having average degree d and fewer than $d^{2+\epsilon}$ vertices has no conjunctive normal form using $\lesssim d^{\epsilon}$ clauses.*

The Conjecture 1.3 can be viewed as a replacement for Conjecture 1.1. The implications for circuit complexity would be the same, as is implicit in Jukna's discussion after his Research Problem 11.17.

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2 Main Argument

We now begin our proof of Theorem 1.2. We start by defining a random conjunctive normal form, designed to have average degree around d with V and W being sets of size $N = d^{10}$. We pick p to be small but independent of d. (Choosing $p = \frac{1}{100}$ would suffice.) Now we define i.i.d. Bernoulli random variables $X_{j,v}$ and $Y_{j,w}$ indexed respectively by $\{1, \dots, n\} \times V$ and $\{1, \dots, n\} \times W$. We set the probability for each of $X_{j,v}$ and $Y_{j,w}$ to be 1 to be p. Now we define

$$A_i = \{v : X_{i,v} = 0\},$$

and

$$B_i = \{w : Y_{i,w} = 0\}.$$

We choose n so that

$$(1 - p^2)^n \sim \frac{d}{N}. \tag{2.1}$$

We achieve Equation 2.1 by picking n to be the nearest integer to $(\frac{1}{p^2}) \ln(\frac{N}{d})$. In particular, this means that n is $O(\log d)$. We let

$$G = \bigcap_{i=1}^{n} CL(A_i, B_i).$$

We will show that after a little pruning, we can modify G to have no $K_{2,2}$'s and still have average degree of at least $d^{1-\epsilon}$.

We now investigate the number of $K_{2,2}$'s in the graph G.

Lemma 2.1. *Let G be as above. Let $v_1, v_2 \in V$ distinct and $w_1, w_2 \in W$ distinct. The probability that there is a $K_{2,2}$ in G on the vertices v_1, w_1, v_2, w_2 is at most $\frac{d^{4-\delta}}{N^{4-\delta}}$, where δ is small depending only on p.*

Proof. We observe that v_1, w_1, v_2, w_2 fail to be a $K_{2,2}$ only when there is some j for which one of $(v_1, w_1), (v_1, w_2), (v_2, w_1), (v_2, w_2)$ lies in the product $A_j^c \times B_j^c$. These are independent events for different j. Now using inclusion-exclusion, we easily see that the probability that a $K_{2,2}$ is not ruled out by the jth clause is $1 - 4p^2 + O(p^3)$. (This is because the event that the jth clause forbids two or more edges of a given $K_{2,2}$, requires at least three of the four vertices to be in A_j or B_j. Thus these events have probability $O(p^3)$.) By independence of the clauses, the probability that the given $K_{2,2}$ is not ruled out at all, and thus is in the graph is $(1-4p^2+O(p^3))^n$. Now in light of the definition of n, namely equation 2.1, the lemma is proved. $\square$

The reader should note that it is here that we have seriously used the presence of more than $\log d$ clauses. The lemma doesn't work unless p is small.

We still need to ensure that most vertices of the graph have a lot of degree.

Lemma 2.2. *Let G be as above. Let $\epsilon > 0$ and d sufficiently large. Let $v \in V$. Then the probability that the degree d_v of v is satisfies*

$$d^{1-\epsilon} \lesssim d_v \lesssim d^{1+\epsilon}$$

is at least $\frac{9}{10}$.

We delay the proof of Lemma 2.2 to point out why Lemmas 2.1 and 2.2 imply Theorem 1.2. In light of Lemma 2.2, the expected number of vertices of V having degree $\gtrsim d^{1-\epsilon}$ is at least $\frac{9N}{10}$. Therefore, with probability at least $\frac{4}{5}$, the graph G has at least $\frac{N}{2}$ vertices in V with degree $\gtrsim d^{1-\epsilon}$. (If the probability is more than $\frac{1}{5}$ that $\frac{N}{2}$ vertices have smaller degree, then the expected number of vertices having smaller degree would be at least $\frac{N}{10}$.) On the other hand from lemma 2.1, the expected number of $K_{2,2}$'s is at most $N^{\delta} d^{4-\delta}$ which by picking p sufficiently small is bounded by d^5. Thus with probability $\frac{1}{2}$ there are at most $2d^5$ copies of $K_{2,2}$ in G. Thus there exists an instance of G with $\frac{N}{2}$ vertices of V having degree $\gtrsim d^{1-\epsilon}$ and having at most $2d^5$ copies of $K_{2,2}$. Let V' be the set of vertices having degree $\gtrsim d^{1-\epsilon}$ and not participating in any $K_{2,2}$'s. Define

$$G' = (V', W, E'),$$

where

$$E' = \bigcap_{i=1}^{n} \left(((A_i \cap V') \times W) \cup (V' \times B_i) \right).$$

Then G' satisfies the conclusion of theorem 1.2.

It remains to prove lemma 2.2. This will be a relatively simple application of the Chernoff-Hoeffding bounds. We shall use the following simple form of them.

Proposition 2.3. *Given M i.i.d. Bernoulli variables $X_1, \ldots X_M$, where the probability of $X_j = 1$ being p, then if q is the probability that*

$$|(\sum_{j=1}^{M} X_j) - pM| \geq \mu M,$$

then

$$q \leq 2e^{-2\mu^2 M}.$$

Proposition 2.3 follows from the results in [Hoeff].

Now we investigate the degree of a vertex v in G. We let $W(v)$ be the set of vertices in W which are adjacent to v. By the definition of G, we have that

$$W(v) = \bigcap_{i:v \notin A_i} B_i.$$

In light of proposition 2.3 there is a universal constant C so that with probability $\frac{19}{20}$ we have that

$$pn - C\sqrt{n} \leq |\{i : v \notin A_i\}| \leq pn + C\sqrt{n}.$$

Here we have chosen $\frac{19}{20}$ to be relatively close to 1. We could have made it even closer to 1 by changing C. We denote $m = |\{i : v \notin A_i\}|$ and denote by $i_1, \ldots i_m$ the elements of $\{i : v \notin A_i\}$. From now on, we work in the case

$$pn - C\sqrt{n} \leq m \leq pn + C\sqrt{n}.$$

We name the sizes of the partial intersections

$$d_j = |\bigcap_{l=1}^{j} A_{i_l}|.$$

then d_m is the degree of v. Now, in light of proposition 2.3 we have for d sufficiently large that with probability at least $1 - \frac{1}{20n}$, as long as $d_{j-1} \geq d^{\frac{1}{2}}$, we have that

$$(1 - p - d^{-\frac{1}{6}})d_{j-1} \leq d_j \leq (1 - p + d^{-\frac{1}{6}})d_{j-1}.$$

(We can choose the probability in this way since exponential decay in n is faster than polynomial decay and since d is on the order of a power of n. Here the choice of $d^{\frac{-1}{6}}$ contributes to the smallness of the factor μ in Proposition 2.3) Thus by induction, we see that as long as we are in the case where all these events hold, which has probabiliy at least $\frac{9}{10}$, we have the inequality

$$N(1 - p - d^{-\frac{1}{6}})^{pn + C\sqrt{n}} \leq d_m \leq N(1 - p + d^{-\frac{1}{6}})^{pn - C\sqrt{n}},$$

which for d sufficiently large, we can rewrite as

$$Nd^{-\epsilon}(1-p)^{pn} \leq d_m \leq Nd^{\epsilon}(1-p)^{pn},$$

which in light of equation 2.1 implies the desired result:

$$d^{1-\epsilon} \lesssim d_m \lesssim d^{1+\epsilon}.$$

Here we are using that for p small, we have $(1-p)^p = 1 - p^2 + o(p^2)$ which is readily verified by taking the first two derivatives of the function $f(x) = (1-x)^x$ near $x = 0$ and checking that it is indeed twice differentiable.

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