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THE GAP BETWEEN MONOTONE AND NON-MONOTONE CIRCUIT COMPLEXITY IS EXPONENTIAL

É. TARDOS

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A. A. Razborov has shown that there exists a polynomial time computable monotone Boolean function whose monotone circuit complexity is at least $n^{c \log n}$. We observe that this lower bound can be improved to $\exp(cn^{1/6-o(1)})$. The proof is immediate by combining the Alon—Boppana version of another argument of Razborov with results of Grötschel—Lovász—Schrijver on the Lovász — capacity, ϑ of a graph.

A. A. Razborov [6, 7] recently proved surprising superpolynomial ($n^{c \log n}$) lower bounds for the monotone circuit complexity of the following two properties of an input graph X on v vertices ($n=v^2$ is the number of input bits):

(a) X has a perfect matching,
(b) X has a clique of size $f(v)$ for some simple function $f(v)$.

The lower bound (b) has been improved to a properly exponential function ($\exp(cn^{1/6-o(1)})$) by N. Alon and R. Boppana [1].

It is a conceptual advantage of (a) that the problem considered there is polynomial time solvable and therefore can be computed by a polynomial size non-monotone Boolean circuit, thus establishing a superpolynomial gap between the monotone and non-monotone circuit complexities of monotone Boolean functions.

The aim of this is note to show that the gap is properly exponential. This follows fairly easily from the Alon—Boppana improvement of Razborov's argument for (b), combined with results of Lovász [4] and Grötschel—Lovász—Schrijver [2] on the Shannon — capacity of a graph.

It is easy to see that the argument of Razborov actually applies not only to the clique number $\omega(X)$ but to any graph function $\varphi(X)$ satisfying $\omega(X) \leqq \varphi(X) \leqq \chi(X)$ where $\chi(X)$ denotes the chromatic number. This observation carries over to the Alon—Boppana improvement and yields the following corollary:

Corollary (A. A. Razborov; N. Alon and R. Boppana). Let $\varphi(X)$ be any monotone graph function such that

$$(*) \qquad \omega(X) \leqq \varphi(X) \leqq \chi(X).$$

Then for any function $3 \leqq f(v) \leqq (v/\log v)^{2/3}/4$ the monotone circuit complexity of deciding whether or not $\varphi(X) \leqq f(v)$ is at least $\exp(c \cdot f(v)^{1/2})$.

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Now, in order to justify the claim that the gap is properly exponential, we just have to point out that there exists a polynomial time computable monotone function $\varphi(X)$ satisfying $(*)$.

In his seminal paper on the Shannon-capacity of graphs [4] Lovász introduced the capacity $\vartheta(X)$. The function $\varphi(X)=\vartheta(\overline{X})$, where $\overline{X}$ denotes the complement of X, is a monotone function satisfying $(*)$. Grötschel, Lovász and Schrijver [GLS] gave a polynomial time approximation algorithm for ϑ. That is, given a graph X and a rational number $\varepsilon>0$ the algorithm computes, in polynomial time, a function $g(X, \varepsilon)$ such that

$$\vartheta(X) \leqq g(X, \varepsilon) \leqq \vartheta(X)+\varepsilon.$$

Now, for any $0<\varepsilon<1/2$ the function $\lfloor g(\overline{X}, \varepsilon)\rceil$, where $\lfloor\alpha\rceil$ denotes the integer nearest to the number α, is a polynomial time computable function satisfying $(*)$. But this function might not be monotone. Let us introduce instead the function

$$\varphi(X) = \lfloor g(\overline{X}, v^{-2})+e(X)\cdot v^{-2}\rceil,$$

where $e(X)$ denotes the number of edges in X. $\varphi(X)$ is a polynomial time computable monotone function satisfying $(*)$.

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Éva Tardos

Comp. Sci. Dept.
Eötvös University
Budapest, Hungary

87-2701—Szegedi Nyomda — Felelős vezető: Surányi Tibor igazgató