ON THE REALIZATION OF FUNCTIONS OF LOGICAL ALGEBRA BY FORMULAE OF FINITE CLASSES (FORMULAE OF LIMITED DEPTH) IN THE BASIS &, $\vee$, $^{-}$ *

O.B. LUPANOV

(Moscow)

Among the mathematical problems of cybernetics an important one is the study of the asymptotic properties of the complexity of control systems [1]. The "power" or "productivity" of a certain type of control system can be defined as the complexity of its best realization by means of functions of a certain class. In this paper we shall study the formulae of limited depth in the basis &, $\vee$, $^{-}$, including disjunctive and conjunctive normal forms [2,3] and their generalizations. It will be shown that, for the construction of the asymptotically best formulae of almost all functions of logical algebra, we can confine ourselves to formulae of a depth not greater than 3, i.e. formulae from $A^3_\vee$, or formulae from $A^3_\&$ (disjunctive and conjunctive normal forms have a depth of less than 2), and we shall give a method of their construction.

1. Formulation of the problem and of the results

We shall define by induction a certain special class of formulae, which are the generalization of disjunctive and conjunctive normal forms.

The class $A^0_\vee = A^0_\&$ consists of the formulae

$$x_1,\ x_2,\ \ldots,\ x_n,\ \ldots,\ \bar{x}_1,\ \bar{x}_2,\ \ldots,\ \bar{x}_n,\ \ldots$$

* *Problemy Kibernetiki*, Vol. 6, 5-14, 1961.

The class $A_{\vee}^{k}$ is defined as follows:

1) $A_{\&}^{k-1} \subseteq A_{\vee}^{k}$;

2) if $F_1 \in A_{\vee}^{k}$, $F_2 \in A_{\vee}^{k}$, then $(F_1 \vee F_2) \in A_{\vee}^{k}$;

3) the class $A_{\vee}^{k}$ does not contain any other formulae, apart from those specified by points (1) and (2).

The class $A_{\&}^{k}$ is defined in a dual way [2].

Let furthermore

$$A^{\infty} = \bigcup_{k=0}^{\infty} A_{\vee}^{k}$$

(it is obvious that $\bigcup_{k=0}^{\infty} A_{\vee}^{k} = \bigcup_{k=0}^{\infty} A_{\&}^{k}$).

Essentially $A_{\vee}^{2}$ is the class of disjunctive normal forms, and $A_{\&}^{2}$ is the class of conjunctive normal forms, $A_{\vee}^{3}$ is the class of "sum of products of sums" of variables or of their negations [4] etc.

For example*

$$(((x_1 \& x_2) \vee (\bar{x}_1 \& x_3)) \vee x_4) \in A_{\vee}^{2};$$

$$((((x_1 \vee (x_2 \& x_3)) \& x_4) \vee \bar{x}_5) \& (\bar{x}_1 \vee \bar{x}_2)) \in A_{\&}^{5}.$$

Let us introduce in the usual way the Shannon function [5, 6]: $L(F)$ is the number of symbols of variables in formulae F;

$$L_{\xi}^{k}(f) = \min L(F)$$

(the minimum is taken with respect to all formulae of the class A_{ξ}^{k}, which realize the function f, if the realization is

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* In the generally accepted form, where the conjunction signs and certain brackets are omitted, these formulae are written as follows:

$$x_1 x_2 \vee \bar{x}_1 x_3 \vee x_4; \quad ((x_1 \vee x_2 x_3)\, x_4 \vee \bar{x}_5)\, (\bar{x}_1 \vee \bar{x}_2)$$

possible*); here $k = 0, 1, \ldots, \infty$; ξ is $\vee$ or is an empty symbol;

$$L_\xi^k(n) = \max L_\xi^k(f)$$

(the maximum is taken over all functions f of the arguments $x_1, \ldots, x_n$).

Because of the principle of duality [2]

$$L_\vee^k(n) = L_\&^k(n);$$

this general value will be denoted by $L^k(n)$. It is easy to see that

$$L^2(n) \geqslant L^3(n) \geqslant \ldots \geqslant L^k(n) \geqslant \ldots \geqslant L^\infty(n). \tag{1}$$

In a paper by Riordan and Shannon [5] it is shown (in terms of series-parallel relay circuits) that**

$$L^\infty(n) \gtrsim \frac{2^n}{\log n}, \tag{2}$$

while for any $\varepsilon > 0$ the proportion of functions f of n arguments for which

$$L^\infty(f) \leqslant (1-\varepsilon)\frac{2^n}{\log n},$$

approaches to zero with increasing n.

Earlier, Shannon [7] has shown that

$$L^\infty(n) \leqslant 3\cdot 2^{n-1} - 2.$$

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* Here and in what follows the word "function" means a function of logical algebra [3] (we shall not distinguish between functions which are obtained from each other by adding or taking away inessential arguments); and the word "collection" (unless otherwise stated) is a finite sequence of zeros and ones (in practice this is a set of values of arguments in functions of logical algebra).

** The symbol $\alpha(n) \gtrsim \beta(n)$ (respectively $\alpha(n) \lesssim \beta(n)$) means that $\underline{\lim}\frac{\alpha(n)}{\beta(n)} \geqslant 1$ (respectively $\overline{\lim}\frac{\alpha(n)}{\beta(n)} \leqslant 1$). Hereafter the symbol log means a logarith of base 2.

In papers [8,9] the author has established that

$$L^{\infty}(n) \sim \frac{2^n}{\log n}, \tag{3}$$

and this statement has been obtained as a consequence of one general theorem.

The basic result of the present paper is the following:

Theorem. When $k \geqslant 3$

$$L^k(n) \sim \frac{2^n}{\log n},$$

while for any $\varepsilon > 0$ the proportion of functions f of n arguments for which

$$\min(L^k_{\vee}(f),\ L^k_{\&}(f)) \leqslant (1-\varepsilon)\frac{2^n}{\log n},$$

approaches to zero with increasing n.

This result is in a certain sense final, since, as is known, when $n \geqslant 2$

$$L^2(n) = n2^{n-1}. \tag{4}$$

In order to prove the theorem, in view of (1) and (2), it is sufficient to show that

$$L^3(n) \leqslant \frac{2^n}{\log n}. \tag{5}$$

This will be done in Section 4 by indicating a suitable synthesis method (note that in view of (1) and (2) this also proves relationship (3)). This method enables us to construct the asymptotically best formulae for almost all functions.

For the sake of completeness we shall also give in this paper a proof of relationships (2) and (4); the first of these is based on a more approximate estimate of the number of formulae than is used in [5] and therefore it is easier to prove.

2. Formulae of the second class

The formulae from $A^2_{\&}$, obtained from the formulae $F_1, \ldots, F_s$ of $A^1_{\vee}$ by means of the operation & (and of parentheses), will be written in the abbreviated form

$$\mathop{\&}_{i=1}^{s} F_i.$$

Lemma 1. Every function of n arguments $x_1, \ldots, x_n$ $(n \geqslant 2)$ can be realized by the formula

$$\mathop{\&}_{i=1}^{s} F_i$$

from $A^2_{\&}$, $F_i \in A^1_{\vee}$, while

$$s \leqslant 2^{n-1}, \; L(F_i) \leqslant n \qquad (1 \leqslant i \leqslant s).$$

Proof. When $n = 2$ the assertion of the lemma is obvious: each function of the arguments x_1, x_2 is realized by one of the following formulae (here $\sigma_1 = 0, 1$; $\sigma_2 = 0, 1$):*

$$(x_1 \& \bar{x}_1), \; (x_1^{\sigma_1} \& x_2^{\sigma_2}), \; x_1^{\sigma_1}, \; x_2^{\sigma_2}, \; ((x_1 \vee x_2^{\sigma_2}) \& (\bar{x}_1 \vee x_2^{\bar{\sigma}_2})),$$
$$(x_1^{\sigma_1} \vee x_2^{\sigma_2}), \; (x_1 \vee \bar{x}_1).$$

Let the statement be proved for functions of the arguments $x_1, \ldots, x_{n-1}$, and let f be an arbitrary function of n arguments. We have

$$f(x_1, \ldots, x_{n-1}, x_n) =$$
$$= (x_n \vee f(x_1, \ldots, x_{n-1}, 0)) \& (\bar{x}_n \vee f(x_1, \ldots, x_{n-1}, 1)).$$

According to the inductive premise the function $f(x_1, \ldots, x_{n-1}, \sigma)$, $\sigma = 0, 1$ can be realized by the formula $\mathop{\&}_{i=1}^{s_\sigma} F_i^\sigma$, where $F_i^\sigma \in A^1_{\vee}$, $s_\sigma \leqslant 2^{n-2}$, $L(F_i^\sigma) \leqslant n - 1$. Then the formula

$$((\mathop{\&}_{i=1}^{s_0} (F_i^0 \vee x_n)) \& (\mathop{\&}_{i=1}^{s_1} (F_i^1 \vee \bar{x}_n)))$$

realizes the function f, and the assertion in the lemma is valid for this formula.

Theorem. When $n \geqslant 2$

$$L^2(n) = n2^{n-1}.$$

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* The sumbol x^σ denoted x when $\sigma = 1$ and $\bar{x}$ when $\sigma = 0$.

Proof. The upper estimate follows from Lemma 1.

The lower estimate. The function

$$f = x_1 + x_2 + \ldots + x_n \pmod 2$$

becomes zero for those, and only those, collections, which have an even number of ones (the number of these collections is equal to 2^{n-1}). If the value of any variable in function f changes, then the value of the function itself also changes. Therefore each sub-formula from F_i from $A^1_{\vee}$ in the formula $\underset{i=1}{\overset{s}{\&}} F_i$ from $A^2_{\&}$, realizing f, contains the symbols of all n variables (i.e. $L(F_i) \geqslant n$), since in the opposite case F_i (and therefore f) would become zero for a pair of collections, which differ in one variable. But then F_i becomes zero for exactly one collection, and therefore $s \geqslant 2^{n-1}$.

3. The lower estimate for $L^{\infty}(n)$

Let $Q_{n,k}$ be the number of formulae from A^{∞} containing not more than k symbols of the variables $x_1, \ldots, x_n$ and which do not contain symbols of other variables.

Lemma 2.

$$Q_{n,k} \leqslant (Cn)^k,$$

where C is a constant.

Proof. It is easy to see that each of the formulae examined, containing l, $l \leqslant k$ symbols of variables, contains $l - 1$ symbols of the operations $\&$, $\vee$, $l - 1$ left-hand brackets and $l - 1$ right-hand brackets. The total number of symbols is $4l - 3$. By writing down on the right-hand side of it $4k - 4l + 3$ times the symbol a, we obtain a word of length $4k$ in the alphabet

$$\{(\ ,\ ),\ \&,\ \vee,\ a,\ x_1, \ldots, x_n,\ \bar{x}_1, \ldots, \bar{x}_n\},$$

containing not more than k letters of the alphabet

$$X = \{x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n\}.$$

Therefore

$$Q_{n,k} \leqslant \sum_{l=1}^{k} C^l_{4k}\, 5^{4k-l} (2n)^l \leqslant C'^k \sum_{l=1}^{n} n^l < (Cn)^k$$

(the places for the letters of X can be selected in not more than C^l_{4k} ways; in these places the letters of X can be substituted in $(2n)^l$ ways; in the remaining $4k - l$ places the letters (,), & $\vee$, a can be arranged in not more than 5^{4k-l} ways).

Theorem (Riordan and Shannon)

$$L^{\infty}(n) \geqslant \frac{2^n}{\log n},$$

while for any $\varepsilon > 0$ the proportion of functions f of the arguments $x_1, \ldots, x_n$ for which

$$L^{\infty}(f) \leqslant (1-\varepsilon)\frac{2^n}{\log n},$$

approaches to zero with increasing n.

Proof. The first assertion follows from the second one. Let us prove the second assertion. If $k \leqslant (1-\varepsilon)\frac{2^n}{\log n}$, then in view of Lemma 2

$$\log\left(\frac{Q_{n,k}}{2^{2^n}}\right) \leqslant (1-\varepsilon)\frac{2^n}{\log n}(\log n+\gamma)-2^n = -\varepsilon 2^n + (1-\varepsilon)\gamma\frac{2^n}{\log n}$$

approaches to k $- \infty$ with increasing n (here $\gamma = \log C$). This proves the theorem, since the number of functions of the arguments $x_1, \ldots, x_n$ is asymptotically equal to 2^{2^n}.*

4. Formulae of the third class

The set of collections

$$(\bar{\alpha}_1, \alpha_2, \ldots, \alpha_d), (\alpha_1, \bar{\alpha}_2, \alpha_3, \ldots, \alpha_d), \ldots, (\alpha_1, \ldots, \alpha_{d-1}, \bar{\alpha}_d)$$

will be called a *sphere* with the centre $(\alpha_1, \ldots, \alpha_d)$. The characteristic function of this sphere can be represented in the form

$$\varphi(x_1, \ldots, x_d) = \bigvee_{j=1}^{d} x_1^{\alpha_1} \ldots x_{j-1}^{\alpha_{j-1}} x_j^{\bar{\alpha}_j} x_{j+1}^{\alpha_{j+1}} \ldots x_d^{\alpha_d}.$$

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* Let us remember that we do not distinguish functions, which are obtained from each other by the addition or elimination of inessential variables.

Therefore

$$\varphi(x_1, \ldots, x_d)\, x_j^{\bar{\alpha}_j} = x_1^{\alpha_1} \ldots x_{j-1}^{\alpha_{j-1}} x_j^{\bar{\alpha}_j} x_{j+1}^{\alpha_{j+1}} \ldots x_d^{\alpha_d} \quad (1 \leqslant j \leqslant d). \tag{6}$$

In addition, φ can be expressed as follows:

$$\varphi = \psi\chi,$$

where

$$\left.\begin{aligned} \psi &= x_1^{\bar{\alpha}_1} \vee \ldots \vee x_d^{\bar{\alpha}_d}, \\ \chi &= \mathop{\&}_{1 \leqslant j_1 < j_2 \leqslant d} \left(x_{j_1}^{\alpha_{j_1}} \vee x_{j_2}^{\alpha_{j_2}}\right). \end{aligned}\right\} \tag{7}$$

In fact, at the centre of the sphere the function ψ becomes zero; over the collections of the sphere both ψ and χ become 1; over each collection, which is different from the centre of the sphere in two or more digits, some conjunctive term in χ becomes zero.

It follows from (7) that

$$L^2_{\&}(\varphi) \leqslant d^2. \tag{8}$$

Lemma 3. If $d = 2^D$, then the set of all collections of length d is divided into $2^d/d$ pairwise non-intersecting spheres.

The proof which is based on the properties of the Hamming code [10] is given in (9), page 69-70 (Lemma 5).

Theorem.

$$L^3(n) \lesssim \frac{2^n}{\log n}.$$

Proof. Let $f(x_1, \ldots, x_n)$ be an arbitrary function. We denote the collections of arguments

$$(x_1, \ldots, x_a),\ (x_{a+1}, \ldots, x_{a+b}),\ (x_{a+b+1}, \ldots, x_{a+b+c}),$$
$$(x_{a+b+c+1}, \ldots, x_{a+b+c+d})$$

respectively by $\tilde{x}$, $\tilde{y}$, $\tilde{z}$, $\tilde{u}$ ($a + b + c + d = n$; $d = 2^D$).

Consider the division of all collections of length d into pairwise non-intersecting spheres (which exist in view of Lemma 3). Let

$$\varphi_i(\tilde{u}), \quad 1 \leqslant i \leqslant \frac{2^d}{d},$$

be the characteristic functions of these spheres. Let

$$f_{i,\tilde{\sigma},\tilde{\varrho}}(\tilde{z},\tilde{u}) = \varphi_i(\tilde{u})\, f(\tilde{\sigma},\tilde{\varrho},\tilde{z},\tilde{u}).$$

We shall denote by $K_{\tilde{\beta}}(\tilde{v})$ the conjunction $v_1^{\beta_1}\dots v_m^{\beta_m}$ ($\tilde{v}=(v_1,\dots,v_m)$ is the collection of arguments, $\tilde{\beta}=(\beta_1,\dots,\beta_m)$ is the collection of their values). Decomposing the functions $\varphi_i(\tilde{u})$ $(\tilde{x},\tilde{y},\tilde{z},\tilde{u})$ with respect to the arguments $\tilde{x}$, $\tilde{y}$, we have

$$\varphi_i(\tilde{u})\, f(\tilde{x},\tilde{y},\tilde{z},\tilde{u}) = \bigvee_{\tilde{\sigma},\tilde{\varrho}} K_{\tilde{\sigma}}(\tilde{x})\, K_{\tilde{\varrho}}(\tilde{y})\, f_{i,\tilde{\sigma},\tilde{\varrho}}(\tilde{z},\tilde{u}).$$

Therefore

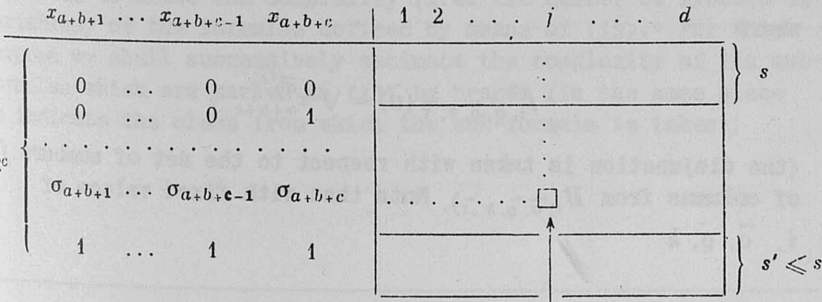
$$f(\tilde{x},\tilde{y},\tilde{z},\tilde{u}) = \bigvee_{i,\tilde{\sigma},\tilde{\varrho}} K_{\tilde{\sigma}}(\tilde{x})\, K_{\tilde{\varrho}}(\tilde{y})\, f_{i,\tilde{\sigma},\tilde{\varrho}}(\tilde{z},\tilde{u}). \qquad (9)$$

The function $f_{i,\tilde{\sigma},\tilde{\varrho}}(\tilde{z},\tilde{u})$ is different from zero only over collections of the form

$$(\sigma_{a+b+1},\dots,\sigma_{a+b+c},\alpha_1^{(i)},\dots,\alpha_{j-1}^{(i)},\bar{\alpha}_j^{(i)},\alpha_{j+1}^{(i)},\dots,\alpha_d^{(i)})$$

(here σ_{a+b+l} is zero or one when $1 \leqslant t \leqslant c$; $(\alpha_1^{(i)},\dots,\alpha_d^{(i)})$ is the centre of the first spehere, $1 \leqslant j \leqslant d$) and therefore it can be specified by Table 1. The rows of the table correspond to the collections $(\sigma_{a+b+1},\dots,\sigma_{a+b+c})$, the columns to the collections of the i^{th} sphere (d items). Let us divide the rows of the matrix M, which defines the value of the function, into bands $A_1,\dots,A_N$ (with s rows in each, and the last band may contain a smaller number of rows:

TABLE 1.



$$f_{i,\tilde{\sigma},\tilde{\varrho}}(\sigma_{a+b+1},\dots,\sigma_{a+b+c},\alpha_1^{(i)},\dots,\alpha_{j-1}^{(i)},\bar{\alpha}_j^{(i)},\alpha_{j+1}^{(i)},\dots,\alpha_d^{(i)}).$$

It is obvious that

$$N \leqslant \frac{2^c}{s} + 1. \tag{10}$$

Let $f_{i,\tilde{\sigma},\tilde{\varrho},k}$ be a function which coincides with $f_{i,\tilde{\sigma},\tilde{\varrho}}$ over the band A_k in M, and equal to zero outside this band. The columns of the matrix corresponding to the function $f_{i,\tilde{\sigma},\tilde{\varrho},k}$ are divided into groups of equal number of columns.

Let $f_{i,\tilde{\sigma},\tilde{\varrho},k,\tilde{\tau}}$ be a function coinciding with $f_{i,\tilde{\sigma},\tilde{\varrho},k}$ over the group of columns $B_{i,\tilde{\sigma},\tilde{\varrho},k,\tilde{\tau}}$, which are equal to $\tilde{\tau}$ (in the band A_k) and equal to zero in the remaining cases. The matrix for the function $f_{i,\tilde{\sigma},\tilde{\varrho},k,\tilde{\tau}}$ has columns of two kinds:

a) columns from $B_{i,\tilde{\sigma},\tilde{\varrho},k,\tilde{\tau}}$, equal to $\tilde{\tau}$ in the band A_k and consisting of zeros outside A_k;

b) the remaining columns consisting of zeros only.

Therefore, the function can be represented in the form of a conjunction of two functions, depending respectively on $\tilde{u}$ and $\tilde{z}$ (see Fig. 1):

1) the functions $f^{(1)}_{i,\tilde{\sigma},\tilde{\varrho},k,\tilde{\tau}}$, which are equal to 1 over the columns of $B_{i,\tilde{\sigma},\tilde{\varrho},k,\tilde{\tau}}$, and equal to zero over the remaining columns;

2) the functions $f^{(2)}_{k,\tilde{\tau}}$, in the matrix of which all columns are equal to $\tilde{\tau}$ in the band A_k, and consist of zeros outside A_k.

In view of (6)

$$f^{(1)}_{i,\tilde{\sigma},\tilde{\varrho},k,\tilde{\tau}}(\tilde{u}) = \varphi_i(\tilde{u})\, f^{(3)}_{i,\tilde{\sigma},\tilde{\varrho},k,\tilde{\tau}}(\tilde{u}),$$

where

$$f^{(3)}_{i,\tilde{\sigma},\tilde{\varrho},k,\tilde{\tau}}(\tilde{u}) = \bigvee x^{\overline{\alpha}_j^{(i)}}_{a+b+c+j}$$

(the disjunction is taken with respect to the set of numbers j of columns from $B_{i,\tilde{\sigma},\tilde{\varrho},k,\tilde{\tau}}$). Note that with fixed values of i, $\tilde{\sigma}$, $\tilde{\varrho}$, k

$$\sum_{\tilde{\tau}} L^1_{\vee}(f^{(3)}_{i,\tilde{\sigma},\tilde{\varrho},k,\tilde{\tau}}) = d. \tag{11}$$

Thus, (taking into account (9)), the function f can be represented in the form

$$f(\tilde{x},\tilde{y},\tilde{z},\tilde{u})=\bigvee_{i,\tilde{\sigma},\tilde{\varrho},k,\tilde{\tau}} K_{\tilde{\sigma}}(\tilde{x})K_{\tilde{\varrho}}(\tilde{y})\varphi_i(\tilde{u})f^{(2)}_{k,\tilde{\tau}}(\tilde{z})f^{(3)}_{i,\tilde{\sigma},\tilde{\varrho},k,\tilde{\tau}}(\tilde{u})=$$
$$=\bigvee_{i,\tilde{\sigma},k,\tilde{\tau}} K_{\tilde{\sigma}}(\tilde{x})\varphi_i(\tilde{u})f^{(2)}_{k,\tilde{\tau}}(\tilde{z})\Big(\bigvee_{\tilde{\varrho}} K_{\tilde{\varrho}}(\tilde{y})f^{(3)}_{i,\tilde{\sigma},\tilde{\varrho},k,\tilde{\tau}}(\tilde{u})\Big). \qquad (12)$$

We shall denote by $D_{\tilde{\beta}'}(\tilde{v})$ the negation of the conjunction $K_{\tilde{\beta}}(\tilde{v})$, i.e. the disjunction $v_1^{\bar{\beta}_1}\vee\ldots\vee v_m^{\bar{\beta}_m}$.

Since with any $\tilde{\varrho}_0$

$$\bigvee_{\tilde{\varrho}} K_{\tilde{\varrho}}(\tilde{\varrho}_0)f^{(3)}_{i,\tilde{\sigma},\tilde{\varrho},k,\tilde{\tau}}(\tilde{u})=\mathop{\&}_{\tilde{\varrho}}\big(D_{\tilde{\varrho}'}(\tilde{\varrho}_0)\vee f^{(3)}_{i,\tilde{\sigma},\tilde{\varrho},k,\tilde{\tau}}(\tilde{u})\big),$$

therefore from (12) we have

$$f(\tilde{x},\tilde{y},\tilde{z},\tilde{u})= \qquad (13)$$
$$=\underbrace{\bigvee_{i,\tilde{\sigma},k}\underbrace{\bigvee_{\tilde{\tau}}\underbrace{\underbrace{K_{\tilde{\sigma}}(\tilde{x})}_{F^7;\,A^1_\&}\;\underbrace{\varphi_i(\tilde{u})}_{F^6;\,A^2_\&}\;\underbrace{f^{(2)}_{k,\tilde{\tau}}(\tilde{z})}_{F^5;\,A^2_\&}\;\underbrace{\Big(\mathop{\&}_{\tilde{\varrho}}\underbrace{\big(\underbrace{D_{\tilde{\varrho}'}(\tilde{y})}_{F^2;\,A^1_\vee}\vee\underbrace{f^{(3)}_{i,\tilde{\sigma},\tilde{\varrho},k,\tilde{\tau}}(\tilde{u})}_{F^1_{i,\tilde{\sigma},\tilde{\varrho},k,\tilde{\tau}};\,A^1_\vee}\big)}_{F^3_{i,\tilde{\sigma},\tilde{\varrho},k,\tilde{\tau}};\,A^1_\vee}\Big)}_{F^4_{i,\tilde{\sigma},k,\tilde{\tau}};\,A^2_\&}}_{F^8;\,A^2_\&}}_{F^9;\,A^3_\vee}}_{F^{10};\,A^3_\vee}.$$

Let us estimate the complexity (i.e. the number of symbols of variables) of the formulae defined by means of (13).* For this purpose we shall successively estimate the complexity of its sub-formulae which are marked in (13) by braces (in the same place we indicate the class from which the sub-formula is taken);

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* The general form corresponding to parallel-series circuits is shown in Fig. 2.

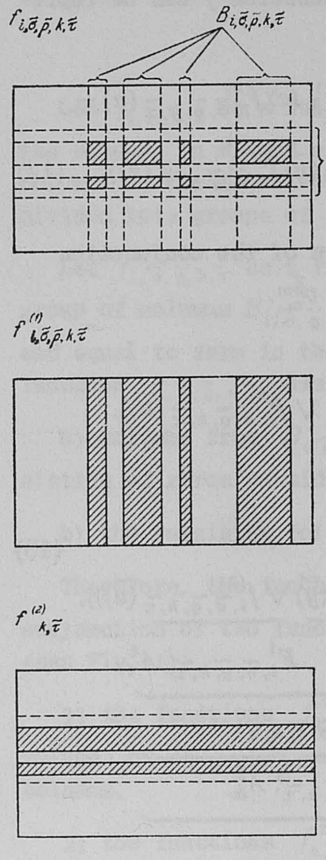

Fig. 1.

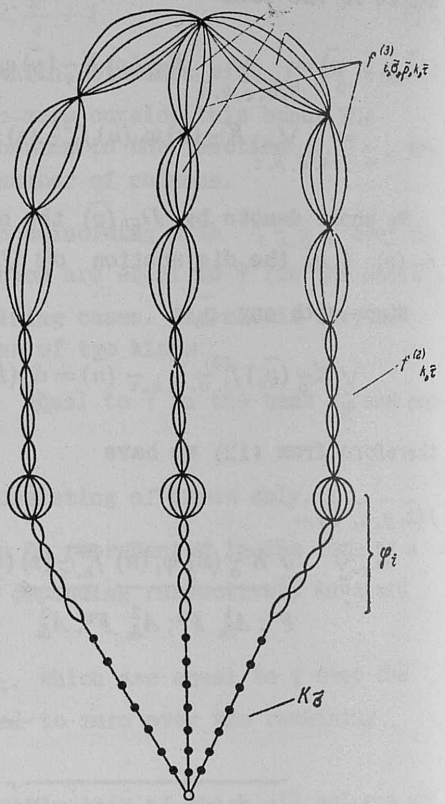

Fig. 2.

$$L(F^2)=b,$$

$$L(F^3_{i,\tilde{\sigma},\tilde{\varrho},k,\tilde{\tau}})=b+L(F^1_{i,\tilde{\sigma},\tilde{\varrho},k,\tilde{\tau}}),$$

$$L(F^4_{i,\tilde{\sigma},k,\tilde{\tau}})=b2^b+\sum_{\tilde{\varrho}} L(F^1_{i,\tilde{\sigma},\tilde{\varrho},k,\tilde{\tau}}),$$

$$L(F^5)\leqslant c2^{c-1} \text{ (see Theorem 2)},$$

$$L(F^6)\leqslant d^2 \text{ (see (8))},$$

$$L(F^7)=a,$$

$$L(F^8)\leqslant a+d^2+c2^c+b2^b+\sum_{\tilde{\varrho}} L(F^1_{i,\tilde{\sigma},\tilde{\varrho},k,\tilde{\tau}}),$$

$$L(F^9) \leqslant 2^s(a+d^2+c2^c+b2^b)+\sum_{\widetilde{\varrho}}\sum_{\widetilde{\tau}} L(F^1_{i,\widetilde{\sigma},\widetilde{\varrho},k,\widetilde{\tau}}) \leqslant$$

$$\leqslant 2^s(a+d^2+c2^c+b2^b)+2^b d$$

(see (11); $\widetilde{\tau}$ takes not more than 2^s values);

$$L^3(n) \leqslant L(F^{10}) \leqslant 2^a \frac{2^d}{d}\left(\frac{2^c}{s}+1\right)(2^s(a+d^2+c2^c+b2^b)+2^b d)$$

(see (10)).

We put

$$d = 2^{[\log n]-1}, \quad c = \lceil 2 \log\log n \rceil, \quad b = [2\log n],$$

$$s = [\log n - 2\log\log n].$$

Then

$$\frac{n}{4} < d \leqslant \frac{n}{2}, \quad 2^b > \frac{n^2}{2}$$

and

$$\frac{b2^{b+s}}{2^b d} \to 0, \quad \frac{c2^{c+s}}{2^b d} \to 0, \quad \frac{d^2 2^s}{2^b d} \to 0, \quad \frac{a2^s}{2^b d} \to 0, \quad \frac{s}{2^c} \to 0.$$

therefore (remembering that $a + b + c + d = n$)

$$L^3(n) \leqslant 2^a \frac{2^d}{d}\frac{2^c}{s}2^b d = \frac{2^n}{s} \sim \frac{2^n}{\log n}.$$

The theorem is proved.

With this the basic theorem formulated in Section 1 is also completely proved.

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A. D. BOOTH

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