Lower Bounds for Constant Weight Codes

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***Abstract*—Let $A(n,2\delta,w)$ denote the maximum number of codewords in any binary code of length n, constant weight w, and Hamming distance 2δ. Several lower bounds for $A(n,2\delta,w)$ are given. For w and δ fixed, $A(n,2\delta,w)\gtrsim n^{w-\delta+1}/w!$ and $A(n,4,w)\sim n^{w-1}/w!$ as $n\to\infty$. In most cases these are better than the "Gilbert bound." Revised tables of $A(n,2\delta,w)$ are given in the range $n<24$ and $\delta<5$.**

I. Lower Bounds for $A(n,4,w)$

Theorem 1:

$$A(n,4,w)\geq\frac{1}{n}\binom{n}{w}.$$

Proof: Let $\mathbb{F}_w^n$ denote the set of $\binom{n}{w}$ binary vectors of length n and weight w, and let $\mathbb{Z}_n=\mathbb{Z}/n\mathbb{Z}$ denote the residue classes modulo n. Consider the map

$$T:\ \mathbb{F}_w^n\to\mathbb{Z}_n$$

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whose value at $\boldsymbol{a}=(a_0,\cdots,a_{n-1})\in\mathbb{F}_w^n$ is

$$T(\boldsymbol{a})=\sum_{a_i=1} i \pmod n$$
$$=\sum_{i=0}^{n-1} ia_i \pmod n. \tag{1}$$

For $0\leq i\leq n-1$ let C_i be the constant weight code $T^{-1}(i)$. We claim that the Hamming distance between any two distinct codewords of C_i, say $\boldsymbol{a}$ and $\boldsymbol{b}$, is at least four. For suppose it is two. Since $\boldsymbol{a}$ and $\boldsymbol{b}$ have weight w this means that $\boldsymbol{a}$ and $\boldsymbol{b}$ agree everywhere except for two positions, one (say the rth) where $\boldsymbol{a}$ is one and $\boldsymbol{b}$ is zero and another (say the sth) where $\boldsymbol{a}$ is zero and $\boldsymbol{b}$ is one. But $T(\boldsymbol{a})=T(\boldsymbol{b})=i$, so from (1)

$$T(\boldsymbol{a})=x+r=i \pmod n,$$
$$T(\boldsymbol{b})=x+s=i \pmod n$$

for some $x\in\mathbb{Z}_n$. This implies $r\equiv s \pmod n$, which is impossible. Thus C_i has a Hamming distance of at least four

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between its codewords. Also

$$|C_0|+|C_1|+\cdots+|C_{n-1}|=\binom{n}{w},$$

so, for at least one j,

$$|C_j|\geq\frac{1}{n}\binom{n}{w}.$$

This completes the proof of Theorem 1.

Corollary 2: Let C_i be as defined in the proof of Theorem 1. Then

$$A(n,4,w)\geq\max_{0\leq i\leq n-1}|C_i|.$$

This is stronger (though less informative). For example, Theorem 1 gives $A(14,4,6)\geq 215$ while Corollary 2 gives $A(14,4,6)\geq 217$ (see Table I).

Remarks

1) This paper was prompted by our seeing B. Bose and T. R. N. Rao's report [1] on unidirectional codes, where (among other things) it is proved that $A(n,4,w)\geq(n+1)^{-1}\binom{n}{w}$. Our proof of Theorem 1 is almost identical to their proof.

2) Other bounds on $A(n,2\delta,w)$ may be found in S. M. Johnson [2] and in [3] and in the references given in these papers. In particular Johnson showed that

$$A(n,2\delta,w)\leq\frac{\binom{n}{w-\delta+1}}{\binom{w}{w-\delta+1}},$$

which implies

$$A(n,2\delta,w)\lesssim\frac{(\delta-1)!\,n^{w-\delta+1}}{w!} \tag{2}$$

as $n\to\infty$. For $\delta=2$ this reads

$$A(n,4,w)\lesssim\frac{n^{w-1}}{w!}.$$

Combining this with Theorem 1 we have Theorem 3.

Theorem 3:

$$A(n,4,w)\sim\frac{n^{w-1}}{w!}$$

for w fixed, as $n\to\infty$.

II. Lower Bounds on $A(n,2\delta,w)$ Based on $GF(q)^{\delta-1}$

Theorem 4: Let q be a prime power such that $q\geq n$. Then

$$A(n,2\delta,w)\geq\frac{1}{q^{\delta-1}}\binom{n}{w}.$$

Proof: Let $q\geq n$ be a prime power, and let the elements of GF(q) be labeled $\omega_0,\omega_1,\cdots,\omega_{q-1}$. Define a map

$$T:\ \mathbb{F}_w^n\to GF(q)^{\delta-1}$$

by

$$T(\boldsymbol{a})=\begin{bmatrix}T_1(\boldsymbol{a})\\ T_2(\boldsymbol{a})\\ \vdots\\ T_{\delta-1}(\boldsymbol{a})\end{bmatrix},$$

where

$$T_1(\boldsymbol{a})=\sum_{a_i=1}\omega_i,$$

$$T_2(\boldsymbol{a})=\sum_{\substack{i<j\\ a_i=a_j=1}}\omega_i\omega_j,$$

$$T_3(\boldsymbol{a})=\sum_{\substack{i<j<k\\ a_i=a_j=a_k=1}}\omega_i\omega_j\omega_k,$$

$$\cdots$$

For each $(\delta-1)$-tuple

$$\boldsymbol{v}=\begin{bmatrix}v_1\\ v_2\\ \vdots\\ v_{\delta-1}\end{bmatrix}\in GF(q)^{\delta-1},$$

let $C_{\boldsymbol{v}}=T^{-1}(\boldsymbol{v})$. Then for some $\boldsymbol{v}$

$$|C_{\boldsymbol{v}}|\geq\frac{1}{q^{\delta-1}}\binom{n}{w}.$$

It remains to show that $C_{\boldsymbol{v}}$ has a Hamming distance of 2δ. Suppose on the contrary that there are vectors $\boldsymbol{a},\boldsymbol{b}\in C_{\boldsymbol{v}}$ with distance $(\boldsymbol{a},\boldsymbol{b})=2\gamma\leq 2\delta-2$. This means that there are 2γ distinct coordinates $r_1,\cdots,r_\gamma,s_1,\cdots,s_\gamma$ such that

$$\begin{array}{rl} & \quad\;\; r_1\; r_2\quad r_\gamma\qquad s_1\; s_2\quad s_\gamma\\ \boldsymbol{a}= & \cdots\ 1\ 1\ 1\ 1\cdots\ 0\ 0\ 0\ 0\cdots,\\ \boldsymbol{b}= & \cdots\ 0\ 0\ 0\ 0\cdots\ 1\ 1\ 1\ 1\cdots,\end{array}$$

and $\boldsymbol{a}$ and $\boldsymbol{b}$ agree in all other coordinates. Write $\alpha_i=\omega_{r_i}$, $\beta_i=\omega_{s_i}$ $(1\leq i\leq\gamma)$. Since $T(\boldsymbol{a})=T(\boldsymbol{b})$ the elementary symmetric functions of the α_i and β_i agree:

$$\sigma_1=\sum_i\alpha_i=\sum_i\beta_i,$$

$$\sigma_2=\sum_{i<j}\alpha_i\alpha_j=\sum_{i<j}\beta_i\beta_j,$$

$$\cdots$$

$$\sigma_{\delta-1}=\sum_{i_1<\cdots<i_{\delta-1}}\alpha_{i_1}\cdots\alpha_{i_{\delta-1}}=\sum_{i_1<\cdots<i_{\delta-1}}\beta_{i_1}\cdots\beta_{i_{\delta-1}}.$$

Therefore $\alpha_1,\cdots,\alpha_\gamma,\beta_1,\cdots,\beta_\gamma$ are 2γ distinct zeros of the polynomial

$$x^\gamma-\sigma_1x^{\gamma-1}+\sigma_2x^{\gamma-2}-\cdots\pm\sigma_\gamma.$$

But a polynomial of degree γ over a field has at most γ zeros. This contradiction completes the proof of Theorem 4.

Again we can strengthen this result.

Corollary 5: Let q be a prime power such that $q\geq n$.

Then

$$A(n,2\delta,w) \geq \max_{v \in GF(q)^{\delta-1}} |C_v|.$$

Remarks

1) For any ϵ there is an $n_0(\epsilon)$ such that for all $n > n_0(\epsilon)$ there is a prime in the interval $(n,(1+\epsilon)n)$ [4, p. 88]. Thus in Theorem 4, q need never be much greater than n and combining this with (2) we have Theorem 6.

Theorem 6:

$$\frac{n^{w-\delta+1}}{w!} \lesssim A(n,2\delta,w) \lesssim \frac{(\delta-1)!\,n^{w-\delta+1}}{w!}$$

for w fixed, as $n \to \infty$.

2) As A. M. Odlyzko has observed, the standard argument used to prove the Gilbert bound for codes (see Berlekamp [5, Theorem 13.71]) when applied to constant weight codes yields Theorem 7.

Theorem 7 (The "Gilbert Bound"):

$$A(n,2\delta,w) \geq \frac{\binom{n}{w}}{\sum_{i=0}^{\delta-1} \binom{w}{i}\binom{n-w}{i}},$$

and so as $n \to \infty$

$$A(n,2\delta,w) \gtrsim \frac{(\delta-1)!\,n^{w-\delta+1}}{w!\binom{w}{\delta-1}}.$$

For small w this is sometimes better than the lower bounds of Theorems 4, 6, and 11. For example when n is large Theorem 7 is stronger than the lower bound of Theorem 6 if w is such that

$$\binom{w}{\delta-1} < (\delta-1)!,$$

but for larger values of w the new bounds are better than the "Gilbert bound."

3) For large n the best upper and lower bounds on $A(n,2\delta,w)$ differ by a factor of

$$\min\left\{(\delta-1)!, \binom{w}{\delta-1}\right\}.$$

In at least one case it is known that the upper bound is correct. From the work of H. Hanani, A. E. Brouwer, and A. Schrijver (the references are given in [3]) it follows that

$$A(n,6,4) \sim \frac{n^2}{12}.$$

III. Lower Bounds on $A(n,2\delta,w)$ Using Sets with Distinct Sums

A subset $S = \{s_1, \cdots, s_n\}$ of $\mathbb{Z}_m$ is called an *S_t-set of size n and modulus m* if all the sums

$$s_{i_1} + s_{i_2} + \cdots + s_{i_t} \tag{3}$$

for $i_1 < i_2 < \cdots < i_t$ are distinct in $\mathbb{Z}_m$.

Provided $t < (n+1)/2$, an S_t-set is automatically an S_u-set for $u < t$. Since there are $\binom{n}{t}$ sums (3), we must have

$$m \geq \binom{n}{t}. \tag{4}$$

The set $\{0,1,2,4\}$ is an example of an S_2-set of size 4 and modulus $m = 6 = \binom{4}{2}$. It can be shown that no S_2-set of size n and modulus $\binom{n}{2}$ exists for $n > 4$; this and other properties of S_2-sets will appear in a companion paper [6].

A perfect difference set is also an S_2-set, for if the *differences* $s_i - s_j$ are distinct then so are the *sums* $s_i + s_j$, but the converse is not true, as the above example shows. The following construction was given by R. C. Bose and S. Chowla [7] in 1962 and generalizes the construction of a Singer perfect difference set (see for example [4, p. 83]).

Theorem 8 (Bose and Chowla): For any prime power q there is an S_t-set of size $q+1$ and modulus $m = (q^{t+1} - 1)/(q-1)$.

Proof: Let $\pi(x)$ be a primitive irreducible polynomial of degree $t+1$ over GF(q) and let ξ be a zero of $\pi(x)$. Then ξ is a primitive element of GF(q^{t+1}),

$$\xi^{q^{t+1}-1} = 1 \qquad \text{and} \qquad \xi^m = \alpha,$$

where α is a primitive element of GF(q). Also the elements of GF(q^{t+1}) may be written as

$$\xi^j = b_0^{(j)} + b_1^{(j)}\xi + \cdots + b_t^{(j)}\xi^t, \tag{5}$$

where $b_i^{(j)} \in GF(q)$, for $0 \leq j \leq q^{t+1} - 2$ (see [8, ch. 4]). Let S consist of those values of j in the range $0 \leq j < m$ for which the coefficients $b_2^{(j)}, \cdots, b_t^{(j)}$ are zero. Then the products

$$\xi^{j_1}\xi^{j_2}\cdots\xi^{j_t}, \qquad j_1 \leq j_2 \leq \cdots \leq j_t,$$

are distinct elements of GF(q^{m+1}) (since these are the products of t linear factors, the representations of these products in the form (5) are all distinct). Therefore S is an S_t set.

Remark: The other construction of S_t-sets given by Bose and Chowla [7, Theorem 1], [4, p. 81, Theorem 3] leads to a bound on $A(n,2\delta,w)$ which is weaker than Theorem 4.

The connection between S_t-sets and $A(n,2\delta,w)$ is given by the following theorem.

Theorem 9: If there exists an $S_{\delta-1}$-set of size n and modulus m then

$$A(n,2\delta,w) \geq \frac{1}{m}\binom{n}{w}.$$

Proof: The proof is similar to that of Theorems 1 and 4, but using the map

$$T: \mathbb{F}_w^n \to \mathbb{Z}_m$$

given by

$$T(\boldsymbol{a}) = \sum_{a_i=1} s_i \pmod m$$

and the codes $C_i = T^{-1}(i)$.

Corollary 10:

$$A(n,2\delta,w) \geq \max_{0 \leq i \leq m-1} |C_i|.$$

From Theorems 8 and 9 we have Theorem 11.

Theorem 11: Let q be the smallest prime power such that $q+1 \geq n$. Then for $\delta \geq 3$

$$A(n,2\delta,w) \geq \frac{q-1}{q^{\delta}-1}\binom{n}{w}.$$

For some values of n this is stronger than Theorem 4, for others, weaker. Asymptotically they are the same.

IV. Tables

Tables of $A(n,2\delta,w)$ for $n \leq 24$ and $\delta \leq 5$ are given in [3] and [8]. A number of the lower bounds for $\delta = 2$ and 3 can now be improved using the above results, and the revised tables are shown in Tables I–IV which appear on the following three pages. The tables for $\delta = 4$ and 5 are included for completeness.

Key to Tables

Unmarked entries are copied from [3].

a) From Theorem 1.
b) From Corollary 2.
c) From Corollary 5.
d) From Corollary 10, using an S_2-set of size 24 and modulus 554 obtained from a perfect difference set [9].
e) From translates of the Nordstrom–Robinson code [10].
f) From the weight distribution of a certain code [10].
g) From a Hadamard matrix [10].
h) See Kibler [11].
i) These values were obtained by Colbourn ([12]; also written communication, August 1979) using the bound given by Johnson in [13, (29)]
j) A. E. Brouwer, [15].

We conclude with some addenda to [3]. Brouwer [10] has communicated to us the following improvements to [5, Table IIIA].

$$T(1,3,6,15,10) = 6,$$
$$T(1,4,6,15,10) = 7,$$
$$T(1,5,6,15,10) = 7,$$
$$T(1,6,6,15,10) = 7 \qquad \text{(not 8)}.$$

The results mentioned in the Note on page 92 of [3] have appeared in Best [14]. In the fifth line of eq. (5), change 197 to 297. On page 89, in line 2 of Section IVA the words "$D(t,k,v)$ where $v =$" are illegible.

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Note Added in Proof

A. E. Brouwer has recently shown that $A(24,10,11) \geq 52$, and P. Delsarte and P. Piret [16] have improved the lower bounds to several values of $A(23,6,w)$ and $A(24,6,w)$.

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TABLE I
A(n,4,w)

n\w	2	3	4	5	6	7	8	9	10	11	12
4	2	1	1								
5	2	2	1	1							
6	3	4	3	1	1						
7	3	7	7	3	1	1					
8	4	8	14	8	4	1	1				
9	4	12	18	18	12	4	1	1			
10	5	13	30	36	30	13	5	1	1		
11	5	17	35	66	66	35	17	5	1	1	
12	6	20	51	[j]75-84	132	75-84	51	20	6	1	1
13	6	26	65	[h]118-132	[j]158-182	158-182	118-132	65	26	6	1
14	7	28	91	[j]169-182	[j]275-308	[j]316-364	275-308	169-182	91	28	7
15	7	35	105	[j]222-271	[j]370-455	[j]582-660	582-660	370-455	222-271	105	35
16	8	37	140	305-336	[j]592-722	[a]715-1040	[j]1164-1320	715-1040	592-722	305-336	140
17	8	44	154-157	424-476	[j]840-952	[h]1224-1753	[h]1496-2210	1496-2210	1224-1753	840-952	424-476
18	9	48	198	[j]504-565	[j]1260-1428	[a]1768-2448	[b]2438-3944	[b]2704-4420	2438-3944	1768-2448	1260-1428
19	9	57	228	612-752	[h]1482-1789	[h]2679-3876	[a]3978-5814	[a]4862-8326	4862-8326	3978-5814	2679-3876
20	10	60	285	816-912	2040-2506	[a]3876-5111	[b]6310-9690	[a]8398-12920	[b]9252-16652	8398-12920	6310-9690
21	10	70	315	1071-1197	2856-3192	[a]5538-7518	[a]9690-13416	[b]14000-22610	[a]16796-27132	16796-27132	14000-22610
22	11	73	385	1386	3927-4389	[a]7752-10032	[b]14550-20674	[a]22610-32794	[b]29414-49742	[a]32066-54264	29414-49742
23	11	83	416-419	1771	5313	[a]10659-14421	[a]21318-28842	[a]35530-52833	[a]49742-75426	[a]58786-104006	58786-104006
24	12	88	498	1859-2011	7084	[a]14421-18216	[b]30667-43263	[b]54484-76912	[b]81752-126799	[a]104006-164565	[b]112720-208012

TABLE II
$A(n,6,w)$

n\w	3	4	5	6	7	8	9	10	11	12
6	2	1	1	1						
7	2	2	1	1	1					
8	2	2	2	1	1	1				
9	3	3	3	3	1	1	1			
10	3	5	6	5	3	1	1	1		
11	3	6	11	11	6	3	1	1	1	
12	4	9	12	22	12	9	4	1	1	1
13	4	13	18	26	26	18	13	4	1	1
14	4	14	28	42	42-51	42	28	14	4	1
15	5	15	42	70	60-88	60-88	70	42	15	5
16	5	20	48	112	90-156	120-150	90-156	112	48	20
17	5	20	68	112-136	[h]119-240[i]	[h]136-283	136-283	119-240	119-240	68
18	6	22	68-72	144-202	160-349	232-428	249-425	232-428	160-349	144-202
19	6	25	[h]76-33	172-228	228-520	332-734[i]	472-789	472-789	332-734	228-520
20	6	30	[h]84-100	232-276	310-651	492-1107[i]	[e]736-1363	944-1421	736-1363	492-1107
21	7	31	[h]105-126	253-350	465-828	668-1695[i]	1068-2364	1286-2702	1286-2702	1068-2364
22	7	37	132-136	294-462	675-1100	708-2277	1288-3775	1450-4416	1574-5064	1450-4416
23	7	40	147-170	399-521	969-1518	[c]929-3162	[c]1551-5819	[c]2167-7521	[c]2576-7953	2576-7953
24	8	42	168-192	532-680	1368-1786	[d]1341-4554	[d]2379-8432	[d]3560-12186[i]	[d]4530-14682	[d]4903-15906

TABLE III
A(n,8,w)

n\w	4	5	6	7	8	9	10	11	12
8	2	1	1	1	1				
9	2	2	1	1	1	1			
10	2	2	2	1	1	1	1		
11	2	2	2	2	1	1	1	1	
12	3	3	4	3	3	1	1	1	1
13	3	3	4	4	3	3	1	1	1
14	3	4	7	8	7	4	3	1	1
15	3	6	10	15	15	10	6	3	1
16	4	6	16	16-22	30	16-22	16	6	4
17	4	7	17	21-31	34-35	34-35	21-31	17	7
18	4	9	20-21	33-41	46-63	48-70	46-63	33-41	20-21
19	4	12	28	52-57	78-97	88-122	88-122	78-97	52-57
20	5	16	40	80	130-142	160-215	176-244	160-215	130-142
21	5	21	56	120	210	280-331	336-399	336-399	280-331
22	5	21	77	176	330	280-493[1]	616-659[1]	672-785[1]	616-659
23	5	23	77-80	253	506	400-801[1]	616-1111	1288-1350[1]	1288-1350
24	6	24	77-92	253-274	759	640-1143[1]	960-1639	1288-2231[1]	2576

TABLE IV
A(n,10,w)

n\w	5	6	7	8	9	10	11	12
10	2	1	1	1	1	1		
11	2	2	1	1	1	1	1	
12	2	2	2	1	1	1	1	1
13	2	2	2	2	1	1	1	1
14	2	2	2	2	2	1	1	1
15	3	3	3	3	3	3	1	1
16	3	3	3	4	3	3	3	1
17	3	3	5	6	6	5	3	3
18	3	4	6	9	10	9	6	4
19	3	4	8	12	19	19	12	8
20	4	5	10	17-18	20-24	38	20-24	17-18
21	4	7	13	21-26	21-41	38-49	38-49	21-41
22	4	7	15-19	22-35	22-57	38-74	38-82	38-74
23	4	8	16-23	23-50	23-87	38-117	38-135	38-135
24	4	9	24-27	[h]27-68[h]	23-119	[f]54-171	38-223	[g]46-247