Proposition 13.6. *For every real matrix A,*

$$\frac{\|A\|_{\mathrm{F}}}{\sqrt{\mathrm{rk}(A)}} \leq \|A\| \leq \|A\|_{\mathrm{F}} .$$

Proof. Observe that $\|A\|_{\mathrm{F}}^2$ is equal to the trace, that is, the sum of diagonal elements of the matrix $B = A^\top A$. On the other hand, the trace of any real matrix is equal to the sum of its eigenvalues. Hence, $\|A\|_{\mathrm{F}}^2 = \sum_{i=1}^n \lambda_i$ where $\lambda_1 \geq \ldots \geq \lambda_n$ are the eigenvalues of B. Since B has only $\mathrm{rk}(B) = \mathrm{rk}(A) = r$ non-zero eigenvalues, and since all eigenvalues of B are nonnegative, the largest eigenvalue λ_1 is bounded by $\|A\|_{\mathrm{F}}^2/r \leq \lambda_1 \leq \|A\|_{\mathrm{F}}^2$. It remains to use the fact mentioned above that $\|A\| = \sqrt{\lambda_1}$. □

Let us now see how the linear algebra argument works in concrete situations.

13.2 Graph decompositions

A bipartite clique is a bipartite complete graph $K_{A,B} = (A \cup B, E)$ with $A \cap B = \emptyset$ and $E = A \times B$.

Let $f(n)$ be the smallest number t such that the complete graph K_n on n vertices $1, 2, \ldots, n$ can be decomposed into t edge-disjoint bipartite cliques. It is not difficult to see that $f(n) \leq n-1$. Indeed, it is enough to remove the vertices $1, 2, \ldots, n-1$ one-by-one, together with their incident edges. This gives us a decomposition of K_n into edge-disjoint stars, that is, bipartite cliques K_{A_i,B_i} with $A_i = \{i\}$ and $B_i = \{i+1, \ldots, n\}$, $i = 1, \ldots, n-1$.

This is, however, just one special decomposition and does not exclude better ones. Still, a classical result of Graham and Pollak (1971) says that the trivial decomposition is in fact the best one! This can be shown using linear algebra.

Theorem 13.7. *The edges of K_n cannot be decomposed into fewer than $n-1$ edge-disjoint biartite cliques.*

Proof (due to Trevberg 1982). We consider a more general question: What is the smallest number t such that the sum of products

$$S(x) := \sum_{1 \leq i < j \leq n} x_i x_j$$

in indeterminates $x = (x_1, \ldots, x_n)$ can be written as the sum

$$S(x) = \sum_{i=1}^{t} \Big(\sum_{j \in A_i} x_j\Big) \cdot \Big(\sum_{j \in B_i} x_j\Big) = \sum_{i=1}^{t} L_i(x) \cdot R_i(x)$$

of products-of-sums with $A_i \cap B_i = \emptyset$ for all $i = 1, \ldots, t$? To answer this question, set $T(x) := \sum_{i=1}^n x_i^2$ and observe that

$$\Big(\sum_{i=1}^n x_i\Big)^2 = \sum_{i=1}^n x_i^2 + 2\sum_{i<j} x_i x_j = T(x) + 2S(x)\,,$$

and hence,

$$T(x) = \Big(\sum_{i=1}^n x_i\Big)^2 - 2S(x) = \Big(\sum_{i=1}^n x_i\Big)^2 - 2\cdot\sum_{i=1}^t L_i(x)\cdot R_i(x)\,. \qquad (13.8)$$

Consider now a homogeneous system of $t+1$ linear equations over $\mathbb{R}$:

$$L_1(x) = 0\,,\ \ldots\,,\ L_t(x) = 0\,,\ x_1 + \cdots + x_n = 0$$

and assume that $t \leq n-2$. Then the system has more variables than equations, implying that it must have a solution $x \in \mathbb{R}^n$ with $x \neq \mathbf{0}$. From $\sum_{i=1}^n x_i = 0$ and $L_i(x) = 0$ for all $i = 1, \ldots, t$ it follows that, for this vector x, the right-hand side of (13.8) must be equal to 0. But the left-hand side is not equal to 0, since $x \neq \mathbf{0}$ implies $T(x) = \sum_{i=1}^n x_i^2 \neq 0$. Thus, our assumption that $t \leq n-2$ has led to a contradiction. $\square$

13.3 Inclusion matrices

A celebrated result, due to Razborov (1987), says that the majority function cannot be computed by constant depth circuits of polynomial size, even if we allow unbounded fanin And, Or and Parity functions as gates. This result was obtained in two steps:

(i) show that functions, computable by small circuits, can be approximated by low degree polynomials, and
(ii) prove that the majority function is hard to approximate by such polynomials.

The proof of (i) is probabilistic, and we will present it later (see Lemma 18.11). The proof of (ii) employs the linear algebra argument, and we present it below.

The *k-threshold function* is a boolean function $T_k^n(x_1, \ldots, x_n)$ which outputs 1 if and only if at least k of the bits in the input vector are 1. A boolean function $g(x_1, \ldots, x_n)$ is a polynomial of degree d over $\mathbb{F}_2$ if it can be written as a sum modulo 2 of products of at most d variables.

Lemma 13.8 (Razborov 1987). *Let $n/2 \leq k \leq n$. Every polynomial of degree at most $2k-n-1$ over $\mathbb{F}_2$ differs from the k-threshold function on at least $\binom{n}{k}$ inputs.*