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COMPLEXITY OF LATTICE—CONFIGURATIONS

by

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Introduction

Let $Y=\{y_1, y_2, \ldots, y_r\}$ and $Z=\{z_1, z_2, \ldots, z_s\}$ be finite sets. Let us define the set $\mathcal{R}$ of the discrete rectangles:

$$\mathcal{R} = \{R: R = U\times V;\, U, V \neq \emptyset\,;\, U\subset Y;\, V\subset Z\}$$

where $U\times V$ denotes the Cartesian product of U and V: $U\times V=\{(y, z): y\in U, z\in V\}$. For the power set $\mathcal{P}=2^{(Y\times Z)}=\{P: P\subset Y\times Z\}$ the relation $\mathcal{R}\subset\mathcal{P}$ holds. Let us denote the complexity of the rectangle $R=U\times V\in\mathcal{R}$ by $\pi(R)$, which is defined by $\pi(R)=|U|+|V|$.

Let us extend the complexity function π for $\mathcal{P}$ in the following way: $\pi(P)=\min\{\sum_{R\in\mathcal{Q}}\pi(R): \mathcal{Q}\subset\mathcal{R}, \bigcup_{R\in\mathcal{Q}} R=P\}$ if $P\in\mathcal{P}$. Further definitions: $\pi(r, s)=\max_{P\in\mathcal{P}}\pi(P)$, $\pi(r)=\pi(r, r)$.

These concepts arised in the switching theory (see e.g. [7]), but they have some interest in themselves, too.

The problem is to determine the values of $\pi(P)$ for certain sets $P\in\mathcal{P}$ and the value of $\pi(r, s)$.

Hencefort I shall try to determine the complexity for three types of sets P and I shall give an asymptotic lower bound for $\pi(r)$.

To the solution of the first problem I shall generalize a theorem of KATONA [2], in which he has proved a conjecture of EHRENFEUCHT and MYCIELSKI. Moreover, I draw two corollaries from this generalization, which do not follow from KATONA's theorem. In this case $\pi(P_1)\sim r\cdot\log_2 r$ holds.

In solving the second problem I use the idea of KATONA and SZEMERÉDI from the proof of Th. 1 in [3] and I get that $\pi(P_2)\sim r\cdot\log_2 r$.

The third problem is the complexity of the Hadamard-matrices [1]. This remained an open question. I shall give only the following lower bound: $\pi(P_3)\geqq(r+1)\sqrt{r}$.

The forth problem is the computing of the value of $\pi(r)$. This is also an unsolved problem. Using LUPANOV's Th. 4 in [5] I shall give only the following asymptotic lower bound for $\pi(r)$ without proof: $\pi(r)\gtrsim\frac{r^2}{\log_2 r\cdot\log_2\log_2 r}$. Accordingly the complexity of P_1 and P_2 can not be maximal.

1. Complexity of diagonalless squares

Assume $Y=Z$ and consider the set $P_1=\{p:p=(y_i, y_j),\ i\neq j,\ 1\leqq i, j\leqq m\}$. What is the value of $\pi(P_1)$?

First I shall prove a lemma which is a generalization of KATONA's theorem in [2] and I shall draw two corollaries of it.

LEMMA 1. *Let* $A_1, A_2, \dots, A_m$ *and* $B_1, B_2, \dots, B_m$ *be subsets of* X $(|X|=n)$, *such that* $A_i\cap B_j\neq\emptyset$ *iff* $i\neq j$ $(1\leqq i, j\leqq m)$. *Then*

$$\sum_{i=1}^{m}\binom{|A_i|+|B_i|}{|A_i|}^{-1}\leqq 1 \quad \text{holds.}$$

PROOF. For arbitrary $i\neq j$, $1\leqq i$, $j\leqq m$ there exist distinct $x_k, x_l\in X$ for which $x_k\in A_i\cap B_j$ and $x_l\in A_j\cap B_i$ hold. Then there is no permutation of the elements $x_1, x_2, \dots, x_n\in X$ for which all the elements of A_i could precede the elements of B_i and all the elements of A_j precede the elements of B_j.

There are $\binom{n}{|A_i|+|B_i|}|A_i|!|B_i|!(n-|A_i|-|B_i|)!$ permutations of the elements $x_1, x_2, \dots, x_n\in X$ in which all the elements of A_i precede the elements of B_i. These permutations are different for different i's therefore

$$\sum_{i=1}^{m}\binom{n}{|A_i|+|B_i|}\cdot|A_i|!|B_i|!(n-|A_i|-|B_i|)!\leqq n!$$

holds which proves our lemma.

Take the particular case, when $B_i=A_i^c=X\setminus A_i$. In this case the assumptions of the lemma have the form $A_i\not\subset A_j$ $(i\neq j)$. That is, Lemma 1 gives the following

Corollary 1. (LUBELL—MESHALKIN—YAMAMOTO inequality [4], [6] and [8]) If $A_1, A_2, \dots, A_m$ are subsets of an n-element set, and $A_i\not\subset A_j$ $(i\neq j)$ then

$$\sum_{i=1}^{m}\binom{n}{|A_i|}^{-1}\leqq 1.$$

Before the second corollary I shall prove a simple lemma.

LEMMA 2.

The function $f(x, y) =$

$$=\begin{cases}\dfrac{1-\langle x\rangle-\langle y\rangle}{\binom{[x]+[y]}{[x]}}+\dfrac{\langle y\rangle}{\binom{[x]+\{y\}}{[x]}}+\dfrac{\langle x\rangle}{\binom{\{x\}+[y]}{\{x\}}} & \text{if } \begin{matrix}\langle x\rangle+\langle y\rangle\leqq 1\\ \langle x\rangle\langle y\rangle>0\end{matrix}\\[2ex] \dfrac{1-\prec x\succ-\prec y\succ}{\binom{\{x\}+\{y\}}{\{x\}}}+\dfrac{\prec x\succ}{\binom{[x]+\{y\}}{[x]}}+\dfrac{\prec y\succ}{\binom{\{x\}+[y]}{\{x\}}} & \text{if } 0\leqq\prec x\succ+\prec y\succ<1\end{cases}$$

is convex in the triangle $1\leqq x$, $1\leqq y$, $x+y\leqq n$, where

$$[z]=\max_{\substack{a \text{ integer}\\ a\leqq z}} a \qquad \{z\}=\min_{\substack{a \text{ integer}\\ a\geqq z}} a$$

$$\langle z\rangle = z-[z] \qquad \prec z\succ = \{z\}-z$$

Moreover in case of an arbitrary x_0, $1\leqq x_0\leqq n-1$ $f(x_0, y)$ *is a strictly monotonically decreasing function of* y *in* $[1, n-x_0]$. *Similarly*, $f(x, y_0)$ *is also a decreasing function of* x *in* $[1, n-y_0]$, *where* y_0 *is fixed in* $[1, n-1]$.

PROOF. Since $f(x, y)$ is a linear and continuous extension of $\binom{x+y}{x}^{-1}$, it is sufficient to show that

(i) for integer x_0, $1\leqq x_0\leqq n-1$, $f(x_0, y)$ is a convex function of y, $y\in[1, n-x_0]$
(ii) for integer y_0, $1\leqq y_0\leqq n-1$, $f(x, y_0)$ is a convex function of x, $x\in[1, n-y_0]$
(iii) for integer x_0, y_0, $2\leqq x_0$, $2\leqq y_0$, $x_0+y_0\leqq n$,

$$f(x_0, y_0)-f(x_0, y_0-1)-f(x_0-1, y_0)+f(x_0-1, y_0-1)\geqq 0,$$

because for the plane

$$p(x, y) = f(x_0, y_0)+\frac{f(x_0+\varepsilon, y_0)-f(x_0, y_0)}{\varepsilon}(x-x_0)+$$
$$+\frac{f(x_0, y_0+\varepsilon)-f(x_0, y_0)}{\varepsilon}(y-y_0)\leqq$$
$$\leqq f(x_0, y_0)+f(x, y_0)-f(x_0, y_0)+f(x_0, y)-f(x_0, y_0)$$

holds ($\varepsilon=1$ or -1) from (i) and (ii), and finally from (iii):

$$f(x, y)-p(x, y)\geqq f(x, y)-f(x_0, y)-f(x, y_0)+f(x_0, y_0)\geqq 0$$

is true and equality holds if (x, y) is equal to (x_0, y_0), $(x_0+\varepsilon, y_0)$ or $(x_0, y_0+\varepsilon)$.

(i) $$\frac{x_0!\,y!}{(x_0+y)!}-\frac{x_0!(y-1)!}{(x_0+y-1)!}=\frac{x_0!(y-1)!}{(x_0+y-1)!}\left(\frac{y}{x_0+y}-1\right)=-\frac{x_0!(y-1)!\,x_0}{(x_0+y)!},$$

$$-\frac{x_0!\,y!\,x_0}{(x_0+y+1)!}+\frac{x_0!(y-1)!\,x_0}{(x_0+y)!}=\frac{x_0!(y-1)!\,x_0}{(x_0+y)!}\times$$

$$\times\left(-\frac{y}{x_0+y+1}+1\right)=\frac{(x_0+1)!(y-1)!}{(x_0+y+1)!}x_0 = x_0\binom{x_0+y+1}{x_0+1}^{-1}>0 \quad\text{if}\quad 2\leqq y.$$

(ii) Similarly $y_0\binom{x+y_0+1}{y_0+1}^{-1}>0$ if $2\leqq x$.

(iii) $$\frac{x_0!\,y_0!}{(x_0+y_0)!}-\frac{(x_0-1)!\,y_0!}{(x_0+y_0-1)!}-\frac{x_0!(y_0-1)!}{(x_0+y_0-1)!}+\frac{(x_0-1)!)(y_0-1)!}{(x_0+y_0-2)!}=$$

$$=\frac{(x_0-1)!(y_0-1)!}{(x_0+y_0-2)!}\left(\frac{x_0y_0}{(x_0+y_0)(x_0+y_0-1)}-\frac{x_0}{x_0+y_0-1}-\frac{y_0}{x_0+y_0-1}+1\right)=$$

$$=\frac{x_0y_0-(x_0+y_0)^2+(x_0+y_0)^2-(x_0+y_0)}{\binom{x_0+y_0-2}{x_0-1}(x_0+y_0)(x_0+y_0-1)}=\frac{(x_0-1)(y_0-1)-1}{\binom{x_0+y_0-2}{x_0-1}(x_0+y_0)(x_0+y_0-1)}\geqq 0,$$

if $\begin{matrix}2\leqq x_0\\ 2\leqq y_0\end{matrix}$

From the proof of the convexity you can also see the monotonity of $f(x, y)$. The proof is completed.

Corollary 2. Let $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_m$ be subsets of an n-element set, such that $A_i \cap B_j \neq \emptyset$ iff $i \neq j$. Then

$$m \leqq \left(f\left(\frac{\sum_{i=1}^{m} |A_i|}{m}, \frac{\sum_{i=1}^{m} |B_i|}{m} \right) \right)^{-1} \leqq \left(f\left(\max_{1 \leqq i \leqq m} |A_i|, \max_{1 \leqq i \leqq m} |B_i| \right) \right)^{-1}.$$

PROOF. Using Lemmas 2 and 1, we obtain

$$m \cdot f\left(\max_{1 \leqq i \leqq m} |A_i|, \max_{1 \leqq i \leqq m} |B_i| \right) \leqq mf\left(\frac{\sum_{i=1}^{m} |A_i|}{m}, \frac{\sum_{i=1}^{m} |B_i|}{m} \right) \leqq \sum_{i=1}^{m} f(|A_i|, |B_i|) \leqq 1,$$

which gives our statement.

LEMMA 3. *Let $U_1, U_2, \ldots, U_n$ and $V_1, V_2, \ldots, V_n$ be non-empty subsets of $Y = \{y_1, y_2, \ldots, y_m\}$ such that $P_1 = \bigcup_{k=1}^{n} (U_k \times V_k)$ holds, where*

$$P_1 = \{p : p = (y_i, y_j), i \neq j, 1 \leqq i, j \leqq m\}$$

Then for the unique λ, defined by $m = 1/f\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$: $\sum_{k=1}^{n} (|U_k| + |V_k|) \geqq \lambda \cdot m$ holds, where $f(x, y)$ is defined in Lemma 2.

PROOF. The following two statements are equivalent

(i) $\bigcup_{k=1}^{n} (U_k \times V_k) = P_1 = \{p : p = (y_i, y_j), i \neq j, 1 \leqq i, j \leqq m\}$.

(ii) The sets $A_i = \{k : y_i \in U_k\}$ and $B_i = \{k : y_i \in V_k\}$ $(1 \leqq i \leqq m)$ have the property, that $A_i \cap B_j \neq \emptyset$ iff $i \neq j$.

(i→ii) If $i \neq j$, $1 \leqq i, j \leqq m$ then $p = (y_i, y_j)$ is an element of $P_1 = \bigcup_{k=1}^{n} (U_k \times V_k)$, that is, there exists a k, $1 \leqq k \leqq n$ for which $y_i \in U_k$ and $y_j \in V_k$. Thus $k \in A_i \cap B_j$ and so $A_i \cap B_j \neq \emptyset$. On the other hand, if $A_i \cap B_j \neq \emptyset$, then there exists a k, $1 \leqq k \leqq n$ for which $k \in A_i \cap B_j$ that is $p = (y_i, y_j)$ is an element of $(U_k \times V_k)$ which means that $p = (y_i, y_j) \in P_1$ and so $i \neq j$.

(ii→i) Assume $p = (y_i, y_j) \in P_1$, for some $1 \leqq i, j \leqq m$, $i \neq j$ follows. Thus, there exists a k, $1 \leqq k \leqq n$ for which $k \in A_i \cap B_j$. Then $p = (y_i, y_j)$ is an element of $(U_k \times V_k)$ which means that $p \in \bigcup_{k=1}^{n} (U_k \times V_k)$.

If $p = (y_i, y_j) \in \bigcup_{k=1}^{n} (U_k \times V_k)$ then there exists a k, $1 \leqq k \leqq n$ for which (y_i, y_j) is an element of $(U_k \times V_k)$. Thus $k \in A_i \cap B_j \neq \emptyset$ and so $i \neq j$, that is, $p \in P_1$.

Let us now suppose that $\sum_{k=1}^{n}(|U_k|+|V_k|)<\lambda m$ holds. Since $\sum_{k=1}^{n}(|U_k|+|V_k|)=$ $=\sum_{i=1}^{m}(|A_i|+|B_i|)$ is true, using the notation

$$\mu=\frac{\sum_{k=1}^{n}(|U_k|+|V_k|)}{m},$$

$$m=1/f\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)>1/f\left(\frac{\mu}{2},\frac{\mu}{2}\right)\geqq\left(f\left(\frac{\sum_{i=1}^{m}|A_i|}{m},\frac{\sum_{i=1}^{m}|B_i|}{m}\right)\right)^{-1}\geqq m$$

hold because on the base of Lemma 2 $f(x, y)$ is decreasing in both x and y, just as $f(x, y)$ is convex and symmetric $\left(f(x,y)=f(y,x)\right)$ and so $f\left(\frac{x+y}{2},\frac{x+y}{2}\right)\leqq f(x,y)$. Finally the last inequality follows from Corollary 2. This is a contradiction. The proof is completed.

THEOREM 1. *Put* $Y=Z$, $|Y|=|Z|=\binom{l}{\left[\frac{l}{2}\right]}=m$ *and* $P_1=\{p:p=(y_i,y_j),\ i\neq j,$ $1\leqq i,\ j\leqq m\}$. Then the complexity of $P_1:\pi(P_1)=l.\binom{l}{\left[\frac{l}{2}\right]}$.

PROOF. From Lemma 3 it follows $\pi(P_1)\geqq\lambda m$. This lower bound is the best possible if $\lambda=l$ is an integer. Namely let A_i's be all the $\left[\frac{l}{2}\right]$ element subsets of the set $X=\{1,2,\ldots,l\}$ and $B_i=A_i^c=X\setminus A_i$. Thus, for A_i's and B_i's (ii) of Lemma 3 holds. In this way for

$$U_k=\{y_i:k\in A_i\},\quad V_k=\{y_i:k\in B_i\}$$

(i) $P_1=\bigcup_{k=1}^{l}(U_k\times V_k)$ also holds. Accordingly if $\lambda=l$ then

$$\lambda\cdot m=l\cdot\binom{l}{\left[\frac{l}{2}\right]}=l/f\left(\left[\frac{l}{2}\right],\left\{\frac{l}{2}\right\}\right)=l/f\left(\frac{l}{2},\frac{l}{2}\right)=\sum_{i=1}^{m}(|A_i|+|B_i|)=$$

$$=\sum_{k=1}^{l}(|U_k|+|V_k|)\geqq\pi(P_1)\geqq\lambda\cdot m,\quad\text{and so}\quad\pi(P_1)=\lambda\cdot m.$$

The proof is completeed

2. Complexity of isosceles right triangles

In case of $Y=Z$ and $P_2=\{p:p=(y_i, y_j), i<j, 1\leqq i, j\leqq m\}$ let us compute the value of $\pi(P_2)$, defined in our introduction.

First I shall prove a lemma which uses the idea of Th. 1. in [3] and I shall draw two conclusions of it.

LEMMA 4. *Let $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_m$ be subsets of an n-element set X, such that $A_i\cap B_j\neq\emptyset$ iff $i<j$ $(1\leqq i, j\leqq m)$.*

Then $\sum_{=1}^{m} 2^{-(|A_i|+|B_i|)}\leqq 1$ *holds.*

PROOF. For arbitrary pair $i<j$, $(1\leqq i, j\leqq m)$ $A_i\cap B_j\neq\emptyset$ holds. Then there is no $A\subset X$, for which both $A_i\subset A$, $A\cap B_i=\emptyset$ and $A_j\subset A$, $A\cap B_j=\emptyset$ hold, because otherwise $\emptyset\neq A_i\cap B_j\subset A\cap B_j=\emptyset$ would be true. Since there are $2^{(n-|A_i|-|B_i|)}$ sets A satisfying $A_i\subset A$ and $A\cap B_i=\emptyset$, thus $\sum_{i=1}^{m} 2^{(n-|A_i|-|B_i|)}\leqq 2^n$ holds which gives our lemma.

LEMMA 5. *Let $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_m$ be subsets of an n-element set X such that $A_i\cap B_j\neq\emptyset$ iff $i<j$ $(1\leqq i, j\leqq m)$.*

Then $m\cdot\log_2 m\leqq \sum_{i=1}^{m}(|A_i|+|B_i|)$.

PROOF: Since 2^{-x} is a convex function and by Lemma 4

$$m\cdot 2^{-\sum_{i=1}^{m}\frac{(|A_i|+|B_i|)}{m}}\leqq \sum_{i=1}^{m} 2^{-(|A_i|+|B_i|)}\leqq 1 \text{ hold,}$$

which proves our lemma.

LEMMA 6. *Let $U_1, U_2, \ldots, U_n$ and $V_1, V_2, \ldots, V_n$ be non-empty subsets of $Y=\{y_1, y_2, \ldots, y_m\}$ such that*

$$\bigcup_{k=1}^{n}(U_k\times V_k)=P_2=\{p:p=(y_i, y_j), i<j, 1\leqq i, j\leqq m\}.$$

Then $\sum_{k=1}^{m}(|U_k|+|V_k|)\geqq m\cdot\log_2 m.$

PROOF. The following two statements are equivalent:

(i) $\bigcup_{k=1}^{n}(U_k\times V_k)=P_2=\{p:p=(y_i, y_j), i<j, 1\leqq i, j\leqq m\}$,

(ii) The sets $A_i=\{k:y_i\in U_k\}$ and $B_i=\{k: y_i\in V_k\}$ $(1\leqq i\leqq m)$ have the property, that $A_i\cap B_j\neq\emptyset$ iff $i<j$.

The proof of this equivalence is the same as it was in Lemma 3, the only difference is, that instead of $i\neq j$ you have to think of $i<j$.

Since $\sum_{k=1}^{n}(|U_k|+|V_k|)=\sum_{i=1}^{m}(|A_i|+|B_i|)$ is true, Lemma 5 gives our Lemma.

THEOREM 2. *Put* $Y=Z$, $|Y|=|Z|=2^l=m$ *and* $P_2=\{p:p=(y_i, y_j), i<j, 1\leqq i, j\leqq m\}$. *Then the complexity of* $P_2:\pi(P_2)=l\cdot 2^l$.

PROOF. From Lemma 6 it follows $\pi(P_2)\geqq m\cdot\log_2 m$. This lower bound is the best possible if $\log_2 m=l$ is an integer. Namely let $Y=\{0, 1\}^l$ and $E=\bigcup_{i=0}^{l-1}\{0, 1\}^i$ be. If $\boldsymbol{\varepsilon}=(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_i)\in E$ let

$$W_{\varepsilon 0}=\{\boldsymbol{\delta}\in\{0, 1\}^l: \boldsymbol{\delta}=(\delta_1, \delta_2, \dots, \delta_l), \delta_j=\varepsilon_j \quad (1\leqq j\leqq i), \delta_{i+1}=0\}$$

$$W_{\varepsilon 1}=\{\boldsymbol{\delta}\in\{0, 1\}^l: \boldsymbol{\delta}=(\delta_1, \delta_2, \dots, \delta_l), \delta_j=\varepsilon_j \quad (1\leqq j\leqq i), \delta_{i+1}=1\} \text{ be.}$$

In this case $P_2=\bigcup_{\varepsilon\in E}(W_{\varepsilon 0}\times W_{\varepsilon 1})$ and $\sum_{\varepsilon\in E}|W_{\varepsilon 0}|+|W_{\varepsilon 1}|=l\cdot 2^l$ trivially hold. The proof is completed.

3. Complexity of Hadamard-matrices

Assume now, that $Y=Z=\{0, 1\}^l$ and $P_3=\{p:p=(y_i, y_j), y_i * y_j=0\}$ (where $y_i * y_j\equiv\sum_{k=1}^{l} y_{ik}\cdot y_{jk} \pmod 2$). Let us compute the value $\pi(P_3)$, defined in our introduction. Let us remark that P_3's are the Hadamard-matrices [1].

First I shall state a simple lemma and by means of it *I* shall give only a lower bound for $\pi(P_3)$. We introduce some notations:

If $U, V\subset Y$, $U * V=\{u * v: u\in U, v\in V\}$.

If $W\subset Y$, let $[W]$ be the subspace spanned by W, and finally $\dim W=\dim [W]$.

LEMMA 7. *Let* U, V *be subsets of the vector space* $Y=\{0, 1\}^l$ *satisfying* $U * V=0$. *Then* $\dim U+\dim V\leqq l$ holds.

PROOF. U has $\dim U$ linearly independent vectors $\{u_i\}_{i=1}^{\dim U}$. For arbitrary $x\in\{0, 1\}^l$ and $x * U=0$ the $x * u_i\equiv 0 \pmod 2$ $1\leqq i\leqq\dim U$ system of equations holds, which has (l-$\dim U$) linearly independent solutions by the Cramer rule. The proof is completed.

THEOREM 3. *Put* $Y=Z=\{0, 1\}^l$ *and* $P_3=\{p:p=(y_i, y_j), \ y_i * y_j\equiv 0 \pmod 2\}$ (*where* $y_i * y_j\equiv\sum_{k=1}^{l} y_{ik}\cdot y_{jk} \pmod 2$). *Then the complexity of* P_3:

$$\pi(P_3)\geqq(2^l+1)2^{\frac{l}{2}}.$$

PROOF. Let $U_1, U_2, \dots, U_n$ and $V_1, V_2, \dots, V_n$ be non-empty subsets of Y, such that $P_3=\bigcup_{k=1}^{n}(U_k\times V_k)$. Then $\sum_{k=1}^{n}|U_k|\cdot|V_k|\geqq|P_3|$. Thus the following inequality holds:

$$(1) \qquad \sum_{k=1}^{n}(|U_k|+|V_k|)=\sum_{k=1}^{n}(|U_k|^{-1}+|V_k|^{-1})\cdot|U_k|\cdot|V_k|\geqq$$

$$\geqq\min_{1\leqq k\leqq n}(|U_k|^{-1}+|V_k|^{-1})\sum_{k=1}^{n}|U_k|\cdot|V_k|\geqq\min_{1\leqq k\leqq n}(|U_k|^{-1}+|V_k|^{-1})\cdot|P_3|.$$

On the other hand.

$$(2)\qquad |U_k|^{-1}+|V_k|^{-1} \geqq 2^{-\dim U_k}+2^{-\dim V_k} \geqq 2^{-\dim U_k}+2^{-l+\dim U_k} \geqq 2^{\left(1-\frac{l}{2}\right)}$$

follows from Lemma 7.

Let us prove now

$$(3)\qquad |P_3| = 2^{l-1}(2^l+1) \text{ by induction over } l.$$

If $l=1$, then $|P_3|=3$. Let us suppose that (3) is true for $l-1$. If y_{il} and y_{jl} are given and $y_{il}\cdot y_{jl}=0$ then $\sum_{k=1}^{l} y_{ik}\, y_{jk}\equiv 0 \pmod 2$ in $2^{l-2}(2^{l-1}+1)$ cases by our inductional assumption. If $y_{il}\cdot y_{jl}=1$ holds then $\sum_{k=1}^{l} y_{ik}\, y_{jk}\equiv 0 \pmod 2$ in $2^{2(l-1)}-2^{l-2}(2^{l-1}+1)$ cases. Since $y_{il}\, y_{jl}=0$ in 3 cases thus $|P_3|=3\cdot 2^{l-2}(2^{l-1}+1)+2^{2(l-1)}-2^{l-2}(2^{l-1}+1)=2^{l-1}(2^l+1)$ for l. From the formulas (1)—(3)

$$\sum_{k=1}^{n}(|U_k|+|V_k|) \geqq \min_{1\leqq k\leqq n}(|U_k|^{-1}+|V_k|^{-1})|P_3| \geqq 2^{1-\frac{l}{2}}2^{l-1}(2^l+1).$$

The proof is completed

4. Complexity of the most complex lattice-configurations

I shall formulate an asymptotic lower bound for $\pi(m)$ which is a simple consequence of LUPANOV's Th. 4 in [5].

THEOREM 4. $\pi(2^l) \gtrsim \dfrac{2^{2l}}{l\cdot \log_2 l}$, *where* $a_l \gtrsim b_l$ *iff* $\liminf_{l} \dfrac{a_l}{b_l} \geqq 1$.

Open problems

The problems of Theorems 3 and 4, which are only partially solved, remained open questions:

1. Let

$$Y = \{y_1, y_2, \ldots, y_{2^l}\} = \{0, 1\}^l$$

and

$$P_3 = \{p : p = (y_i, y_j),\ 1 \leqq i, j \leqq 2^l,\ y_i * y_j \equiv 0 \pmod 2\}$$

be where $y_i * y_j \equiv \sum_{k=1}^{l} y_{ik}\cdot y_{jk} \pmod 2$ and let us give a system $U_1, U_2, \ldots, U_n$; $V_1, V_2, \ldots, V_n$ for which $P=\bigcup_{k=1}^{n}(U_k\times V_k)$ and

$$\sum_{k=1}^{n}(|U_k|+|V_k|) = \text{minimal},$$

where $U_k\times V_k$ is the Cartesian product of U_k and V_k.

2. Let $Y=\{y_1, y_2, \ldots, y_m\}$ be and let us denote by $\pi(m)$ the minimal number that for every $P\subset(Y\times Y)$ there exists a system $U_1, U_2, \ldots, U_n; V_1, V_2, \ldots, V_n$ such that

$$P=\bigcup_{k=1}^{n}(U_k\times V_k) \text{ and}$$

$$\pi(m)\geqq\sum_{k=1}^{n}(|U_k|+|V_k|) \text{ hold,}$$

where $(U\times V)$ is the Cartesian product of U and V. What is the exact value of $\pi(m)$?

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