Neighborly families of boxes and bipartite coverings

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Dedicated to Professor Paul Erdős on the occasion of his 80^{th} *birthday*

Abstract

A *bipartite covering of order* k of the complete graph K_n on n vertices is a collection of complete bipartite graphs so that every edge of K_n lies in at least 1 and at most k of them. It is shown that the minimum possible number of subgraphs in such a collection is $\Theta(kn^{1/k})$. This extends a result of Graham and Pollak, answers a question of Felzenbaum and Perles, and has some geometric consequences. The proofs combine combinatorial techniques with some simple linear algebraic tools.

1 Introduction

Paul Erdős taught us that various extremal problems in Combinatorial Geometry are best studied by formulating them as problems in Graph Theory. The celebrated Erdős de Bruijn theorem [3] that asserts that n non-collinear points in the plane determine at least n distinct lines is one of the early examples of this phenomenon. An even earlier example appears in [4] and many additional ones can be found in the surveys [5] and [12]. In the present note we consider another example of an extremal geometric problem which is closely related to a graph theoretic one. Following the Erdős tradition we study the graph theoretic problem in order to deduce the geometric consequences.

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A finite family $\mathcal{C}$ of d-dimensional convex polytopes is called *k-neighborly* if $d - k \leq dim(C \cap C') \leq d - 1$ for every two distinct members C and C' of the family. In particular, a 1-neighborly family is simply called *neighborly.* In this case the dimension of the intersection of each two distinct members of the family is precisely $d - 1$. Neighborly families have been studied by various researchers, see, e.g., [10], [14], [15], [16], [17]. In particular it is known that the maximum possible cardinality of a neighborly family of d-simplices is at least 2^d ([16]) and at most 2^{d+1} ([14]). The maximum possible cardinality of a neighborly family of *standard boxes* in R^d, that is, a neighborly family of d-dimensional boxes with edges parallel to the coordinate axes, is precisely $d+1$. This has been proved by Zaks [17], by reducing the problem to a theorem of Graham and Pollak [8] about bipartite decompositions of complete graphs. In the present note we consider the more general problem of k-neighborly families of standard boxes. The following result determines the asymptotic behaviour of the maximum possible cardinality of such a family.

Theorem 1.1 *For $1 \leq k \leq d$, let $n(k, d)$ denote the maximum possible cardinality of a k-neighborly family of standard boxes in R^d. Then*

$$(i) \qquad d + 1 = n(1, d) \leq n(2, d) \leq \cdots \leq n(d - 1, d) \leq n(d, d) = 2^d.$$

$$(ii) \qquad (\frac{d}{k})^k \leq \prod_{i=0}^{k-1}(\lfloor \frac{d+i}{k} \rfloor + 1) \leq n(k, d) \leq \sum_{i=0}^{k} 2^i \binom{d}{i} < 2(\frac{2ed}{k})^k.$$

This answers a question of Felzenbaum and Perles [6], who asked if for fixed k, $n(k, d)$ is a nonlinear function of d.

As in the special case $k = 1$, the function $n(k, d)$ can be formulated in terms of bipartite coverings of complete graphs. A *bipartite covering* of a graph G is a family of complete bipartite subgraphs of G so that every edge of G belongs to at least one such subgraph. The covering is of *order k* if every edge lies in at most k such subgraphs. The *size* of the covering is the number of bipartite subgraphs in it. The following simple statement provides an equivalent formulation of the function $n(k, d)$.

Proposition 1.2 *For $1 \leq k \leq d$, $n(k, d)$ is precisely the maximum number of vertices of a complete graph that admits a bipartite covering of order k and size d.*

The rest of this note is organized as follows. In Section 2 we present the simple proof of Proposition 1.2. The main result, Theorem 1.1, is proved in Section 3. Section 4 contains some possible extensions and open problems.

2 Neighborly families and bipartite coverings

There is a simple one to one correspondence between k-neighborly families of n standard boxes in R^d and bipartite coverings of order k and size d of the complete graph K_n. To see this correspondence, consider a k-neighborly family $\mathcal{C} = \{C_1, \ldots, C_n\}$ of n standard boxes in R^d. Since any two boxes have a nonempty intersection, there is a point in the intersection of all the boxes (by the trivial, one dimensional case of Helly's Theorem, say). By shifting the boxes we may assume that this point is the origin O. If $n \geq 2$, O must lie in the boundary of each box, since it belongs to all boxes, and the dimension of the intersection of each pair of boxes is strictly smaller than d. Put $V = \{1, 2, \ldots, n\}$. For each coordinate x_i, $1 \leq i \leq d$, let H_i be the complete bipartite graph on V whose sets of vertices are $V_i^+ = \{j : C_j$ is contained in the half space $x_i \geq 0\}$, and $V_i^- = \{j : C_j$ is contained in the half space $x_i \leq 0\}$. It is not difficult to see that if the dimension of $C_p \cap C_q$ is $d - r$, then the edge pq of the complete graph on V lies in exactly r of the subgraphs H_i. Therefore, the graphs H_i form a bipartite covering of order k and size d.

Moreover, the above correspondence is invertible; given a bipartite covering of the complete graph on $V = \{1, 2, \ldots, n\}$ by complete bipartite subgraphs $H_1, \ldots, H_d$ one can define a family of n standard boxes as follows. Let V_i^+ and V_i^- denote the two color classes of H_i. For each j, $1 \leq j \leq n$, let C_j be the box defined by the intersection of the unit cube $[-1, 1]^d$ with the half spaces $x_i \geq 0$ for all i for which $j \in V_i^+$ and the half spaces $x_i \leq 0$ for all i for which $j \in V_i^-$. If the given covering is of order k, the family of standard boxes obtained is k-neighborly.

The correspondence above clearly implies the assertion of Proposition 1.2, and enables us to study, in the next section, bipartite coverings, in order to prove Theorem 1.1.

3 Economical bipartite coverings

In this section we prove Theorem 1.1. In view of Proposition 1.2 we prove it for the function $n(k,d)$ that denotes the maximum number of vertices of a complete graph that admits a bipartite covering of order k and size d.

Part (i) of the theorem is essentially known. The fact that $n(1,d) = d+1$ is a Theorem of Graham and Pollak [8], [9]. See also [7], [11], [18], [13], [1] and [2] for various simple proofs and extensions. The statement that for every fixed d, $n(k,d)$ is a non-decreasing function of k is obvious and the claim that $n(d,d) = 2^d$ is very simple. Indeed, the chromatic number of any graph that can be covered by d bipartite subgraphs is at most 2^d, implying that $n(d,d) \leq 2^d$. To see the lower bound, let V be a set of 2^d vertices denoted by all the binary vectors $\epsilon = (\epsilon_1, \ldots, \epsilon_d)$, and let H_i be the complete bipartite graph whose classes of vertices are all the vertices labelled by vectors with $\epsilon_i = 0$ and all the vertices labelled by vectors with $\epsilon_i = 0$. Trivially $H_1, \ldots, H_d$ form a bipartite covering (of order d and size d) of the complete graph on V, showing that $n(d,d) = 2^d$, as claimed.

The lower bound in part (ii) of the theorem is proved by a construction, as follows. For each i, $0 \leq i \leq k-1$, define $d_i = \lfloor (d+i)/k \rfloor$ and $D_i = \{1, 2, \ldots, d_i, d_i+1\}$. Observe that $\sum_{r=0}^{k-1} d_r = d$. Let V denote the set of vectors of length k defined as follows

$$V = \{\ (\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1}) : \epsilon_i \in D_i\ \}.$$

For each r, $0 \leq r \leq k-1$, and each j, $1 \leq j \leq d_r$, let $H_{r,j}$ denote the complete bipartite graph on the classes of vertices

$$A_{r,j} = \{\ (\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1}) : \epsilon_r = j\ \}$$

and

$$B_{r,j} = \{\ (\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1}) : \epsilon_r \geq j+1\ \}.$$

Altogether there are $\sum_{r=0}^{k-1} d_r = d$ bipartite subgraphs $H_{r,j}$. It is not too difficult to see that they form a bipartite covering of the complete graph on V. In fact, if $(\epsilon_0, \ldots, \epsilon_{k-1})$ and $(\epsilon'_0, \ldots, \epsilon'_{k-1})$ are two distinct members of V, and they differ in s coordinates, then the edge joining them lies in precisely s of the bipartite graphs. Since $1 \leq s \leq k$ for each such two members, the above covering is of order k, implying the lower bound in part (ii) of the theorem.

The upper bound in part (ii) is proved by a simple algebraic argument. Let $H_1, \ldots, H_d$ be a bipartite covering of order k and size d of the complete graph on the set of vertices $N = \{1, 2, \ldots, n\}$. Let A_i and B_i denote the two vertex classes of H_i. For each $i \in N$, define a polynomial $P_i = P_i(x_1, \ldots, x_d, y_1, \ldots y_d)$ as follows:

$$P_i = \prod_{j=1}^{k} \Big(\sum_{p:\ i \in A_p} x_p + \sum_{q:\ i \in B_q} y_q - j \Big).$$

For each $i \in N$ let $e_i = (b_{i1}, \ldots, b_{id}, a_{i1}, \ldots, a_{id})$ be the zero-one vector in which $a_{ip} = 1$ if $i \in A_p$ (and $a_{ip} = 0$ otherwise), and, similarly, $b_{iq} = 1$ if $i \in B_q$ (and $b_{iq} = 0$ otherwise). The crucial point is the fact that

$$P_i(e_j) = 0 \quad for \quad all \quad i \neq j \quad and \quad P_i(e_i) \neq 0. \tag{1}$$

This holds as the value of the sum

$$\sum_{p:\ i \in A_p} x_p + \sum_{q:\ i \in B_q} y_q$$

for $x_p = b_{jp}$ and $y_q = a_{jq}$ is precisely the number of bipartite subgraphs in our collection in which i and j lie in distinct color classes. This number is 0 for $i = j$ and is between 1 and k for all $i \neq j$, implying the validity of (1).

Let $\overline{P}_i = \overline{P}_i(x_1, \ldots, x_d, y_1, \ldots, y_d)$ be the multilinear polynomial obtained from the standard representation of P_i as a sum of monomials by replacing each monomial of the form $c \prod_{s \in S} x_s^{\delta_s} \prod_{t \in T} y_t^{\gamma_t}$, where all the δ_s and γ_t are positive, by the monomial $c \prod_{s \in S} x_s \prod_{t \in T} y_t$. Observe that when all the variables x_p, y_q attain $0,1$-values, $P_i(x_1, \ldots, y_d) = \overline{P}_i(x_1, \ldots, y_d)$, since for any positive δ, $0^\delta = 0$ and $1^\delta = 1$. Therefore, by (1),

$$\overline{P}_i(e_j) = 0 \quad for \quad all \quad i \neq j \quad and \quad \overline{P}_i(e_i) \neq 0. \tag{2}$$

By the above equation, the polynomials $\overline{P}_i$ $(i \in N)$ are linearly independent. To see this, suppose this is false, and let

$$\sum_{i \in N} c_i P_i(x_1, \ldots, y_d) = 0,$$

be a nontrivial linear dependence between them. Then there is an $i' \in N$ so that $c_{i'} \neq 0$. By substituting $(x_1, \ldots, y_d) = e_{i'}$ we conclude, by (2), that $c_{i'} = 0$, contradiction. Thus these polynomials are indeed linearly independent. Each polynomial $\overline{P}_i$ is a multilinear polynomial of degree at most k. Moreover, by their definition they do not contain any monomials that contain both x_i and y_i for the same i. It thus follows that all the polynomials $\overline{P}_i$ are in the space generated by all the monomials $\prod_{s \in S} x_s \prod_{t \in T} y_t$, where S and T range over all subsets of N satisfying $|S| + |T| \leq k$ and $S \cap T = \emptyset$. Since there are $m = \sum_{i=0}^{k} 2^i \binom{d}{i}$ such pairs S, T, this is the dimension of the space considered, and as the polynomials $\overline{P}_i$ are n linearly independent members of this space it follows that $n \leq m$. This completes the proof of part (ii) and hence the proof of Theorem 1.1. $\square$

4 Concluding remarks and open problems

The proof of the upper bound for the function $n(k, d)$ described above can be easily extended to the following more general problem. Let K be an arbitrary subset of cardinality k of the set $\{1, 2, \ldots, d\}$. A bipartite covering $H_1, \ldots, H_d$ of size d of the complete graph K_n on n vertices is called a covering of *type* K if for every edge e of K_n, the number of subgraphs H_i that contain e is a member of K. The proof described above can be easily modified to show that the maximum n for which K_n admits a bipartite covering of type K and size d, where $|K| = k$, is at most $\sum_{i=0}^{k} 2^i \binom{d}{i}$. There are several examples of sets K for which one can give a bigger lower bound than the one given in Theorem 1.1 for the special case of $K = \{1, \ldots, k\}$. For example, for $K = \{2, 4\}$, there is a bipartite covering $H_1, \ldots, H_d$ of type K of a complete graph on $n = 1 + \binom{d}{2}$ vertices. To see this, denote the vertices by all subsets of cardinality 0 or 2 of a fixed set D of d elements and define, for each $i \in D$, a complete bipartite graph whose classes of vertices are all subsets that contain i and all subsets that do not contain i. Similar examples exists for types K of bigger cardinality.

One can consider bipartite coverings of prescribed type of other graphs besides the complete graph, and the algebraic approach described above can be used to supply lower bounds for the minimum possible number of bipartite subgraphs in such a cover, as a function of the rank of the adjacency matrix of the graph (and the type).

The main problem that remains open is, of course, that of determining precisely the function $n(k,d)$ for all k and d. Even the precise determination of $n(2,d)$ seems difficult.

References

[1] N. Alon, *Decomposition of the complete r-graph into complete r-partite r-graphs*, Graphs and Combinatorics 2 (1986), 95-100.

[2] N. Alon, R. A. Brualdi and B. L. Shader, *Multicolored forests in bipartite decompositions of graphs*, J. Combinatorial Theory, Ser. B (1991), 143-148.

[3] N. G. de Bruijn and P. Erdős, *On a combinatorial problem*, Indagationes Math. 20 (1948), 421-423.

[4] P. Erdős, *On sequences of integers none of which divides the product of two others, and related problems*, Mitteilungen des Forschungsinstituts für Mat. und Mech., Tomsk, 2 (1938), 74-82.

[5] P. Erdős and G. Purdy, *Some extremal problems in combinatorial geometry*, in: Handbook of Combinatorics (R. L. Graham, M. Grötschel and L. Lovász eds.), North Holland, to appear.

[6] A. Felzenbaum and M. A. Perles, Private communication.

[7] R. L. Graham and L. Lovász, *Distance matrix polynomials of trees*, Advances in Math. 29 (1978), 60-88.

[8] R. L. Graham and H. O. Pollak, *On the addressing problem for loop switching*, Bell Syst. Tech. J. 50 (1971), 2495-2519.

[9] R. L. Graham and H. O. Pollak, *On embedding graphs in squashed cubes*, In: Lecture Notes in Mathematics 303, pp 99-110, Springer Verlag, New York-Berlin-Heidelberg, 1973.

[10] J. Kasem, *Neighborly families of boxes*, Ph. D. Thesis, Hebrew University, Jerusalem, 1985.

[11] L. Lovász, *Combinatorial Problems and Exercises* , Problem 11.22, North Holland, Amsterdam 1979.

[12] J. Pach and P. Agarwal, **Combinatorial Geometry**, DIMACS Tech. Report 41–51, 1991 (to be published by J. Wiley).

[13] G. W. Peck, *A new proof of a theorem of Graham and Pollak*, Discrete Math. 49 (1984), 327-328.

[14] M. A. Perles, *At most* 2^{d+1} *neighborly simplices in* E^d, Annals of Discrete Math. 20 (1984), 253-254.

[15] J. Zaks, *Bounds on neighborly families of convex polytopes*, Geometriae Dedicata 8 (1979), 279-296.

[16] J. Zaks, *Neighborly families of* 2^d *d-simplices in* E^d, Geometriae Dedicata 11 (1981), 505-507.

[17] J. Zaks, Amer. Math. Monthly 92 (1985), 568-571.

[18] H. Tverberg, *On the decomposition of* K_n *into complete bipartite graphs*, J. Graph Theory 6 (1982), 493-494.