

# On Sperner's Theorem

Let  $\mathcal{F}$  be a family of subsets of  $[n]$  with no chain longer than  $k$ . Then

$$|\mathcal{F}| \leq \text{Sum of the largest } k \text{ binomial coefficients in } n.$$

Proof. The claim is trivial for  $n = 1$  and all  $k$ . Assume the claim is true for ground sets of sizes at most  $n$  and take a family  $\mathcal{F}$  of subsets of  $[n + 1]$  having no chain longer than  $k$ . We can write  $\mathcal{F}$  as a disjoint union  $\mathcal{F} = \mathcal{G} \cup \mathcal{H}$ , where  $\mathcal{G} = \{\text{sets in } \mathcal{F} \text{ containing } n + 1\}$  and  $\mathcal{H} = \{\text{sets in } \mathcal{F} \text{ not containing } n + 1\}$ . Consider a subfamily  $\mathcal{M} \subset \mathcal{H}$  which consists of sets which are the top sets of some chain of length  $k$  in  $\mathcal{H}$  and add  $\mathcal{M}$  to  $\mathcal{G}$ . Write  $\mathcal{G}' = \mathcal{G} \cup \mathcal{M}$  and  $\mathcal{H}' = \mathcal{H} / \mathcal{M}$ . Then  $\mathcal{F}$  is a disjoint union of  $\mathcal{G}'$  and  $\mathcal{H}'$ . We have that  $\mathcal{H}'$  has no chain longer than  $k - 1$  as we shortened the maximum length chains. Also, no element in  $\mathcal{M}$  is contained in an element of  $\mathcal{G}$  as in this way we would have a chain longer than  $k + 1$  in  $\mathcal{F}$ . Thus  $\mathcal{G}'$  corresponds to a set system on the ground set  $[n]$  by removing the element  $n + 1$  from elements of  $\mathcal{G}$ . This system has no chain longer than  $k + 1$  elements from  $\mathcal{M}$  can extend a chain in  $\mathcal{G}$  by at most one. Thus, by induction, the size of  $\mathcal{G}'$  is no more than the  $k + 1$  largest binomial coefficients in  $n$  and  $\mathcal{H}'$  has no more elements than the sum of  $k - 1$  largest binomial coefficients in  $n$ . By Pascal's identity this gives in total the sum of  $k$  largest binomial coefficients in  $n + 1$ .