Proof. It is enough to show that $|H \setminus P| < \frac{1}{2}|H|$. For every non-popular point $b \in A_i$, we have that

$$|\{\boldsymbol{a} \in H \;:\; a_i = b\}| < \frac{1}{2k}\frac{|H|}{|A_i|}\,.$$

Since the number of non-popular points in each set A_i does not exceed the total number of points $|A_i|$, we obtain

$$\begin{aligned} |H \setminus P| &\le \sum_{i=1}^{k}\sum_{b\notin P_i} |\{\boldsymbol{a} \in H \;:\; a_i = b\}| < \sum_{i=1}^{k}\sum_{b\notin P_i}\frac{1}{2k}\frac{|H|}{|A_i|} \\ &\le \sum_{i=1}^{k}\frac{1}{2k}|H| = \frac{1}{2}|H|\,. \quad \square \end{aligned}$$

Corollary 2.15. *In any* 2α*-dense* 0-1 *matrix* H *either a* $\sqrt{\alpha}$*-fraction of its rows or a* $\sqrt{\alpha}$*-fraction of its columns (or both) are* $(\alpha/2)$*-dense.*

Proof. Let H be an $m\times n$ matrix. We can view H as a subset of the Cartesian product $[m] \times [n]$, where $(i, j) \in H$ iff the entry in the i-th row and j-th column is 1. We are going to apply Lemma 2.14 with $k = 2$. We know that $|H| \ge 2\alpha mn$. So, if P_1 is the set of all rows with at least $\frac{1}{4}|H|/|A_1| = \alpha n/2$ ones, and P_2 is the set of all columns with at least $\frac{1}{4}|H|/|A_2| = \alpha m/2$ ones, then Lemma 2.14 implies that

$$\frac{|P_1|}{m}\cdot\frac{|P_2|}{n} \ge \frac{1}{2}\frac{|H|}{mn} \ge \frac{1}{2}\cdot\frac{2\alpha mn}{mn} = \alpha\,.$$

Hence, either $|P_1|/m$ or $|P_2|/n$ must be at least $\sqrt{\alpha}$, as claimed. $\square$

2.6 The Lovász–Stein theorem

This theorem was used by Stein (1974) and Lovász (1975) in studying some combinatorial covering problems. The advantage of this result is that it can be used to get existence results for some combinatorial problems using constructive methods rather than probabilistic methods.

Given a family $\mathcal{F}$ of subsets of some finite set X, its *cover number* of $\mathcal{F}$, $\mathrm{Cov}(\mathcal{F})$, is the minimum number of members of $\mathcal{F}$ whose union covers all points (elements) of X.

Theorem 2.16. *If each member of* $\mathcal{F}$ *has at most* a *elements, and each point* $x \in X$ *belongs to at least* v *of the sets in* $\mathcal{F}$*, then*

$$\mathrm{Cov}(\mathcal{F}) \le \frac{|\mathcal{F}|}{v}(1 + \ln a)\,.$$

Proof. Let $N = |X|$, $M = |\mathcal{F}|$ and consider the $N \times M$ 0-1 matrix $A = (a_{x,i})$, where $a_{x,i} = 1$ iff $x \in X$ belongs to the i-th member of $\mathcal{F}$. By our assumption, each row of A has at least v ones and each column at most a ones. By double counting, we have that $Nv \geq Ma$, or equivalently,

$$\frac{M}{v} \leq \frac{N}{a}\,. \tag{2.7}$$

Our goal is to show that then A must contain an $N \times K$ submatrix C with no all-0 rows and such that

$$K \leq N/a + (M/v)\ln a \leq (M/v)(1 + \ln a)\,.$$

We describe a constructive procedure for producing the desired submatrix C. Let $A_a = A$ and define A'_a to be any maximal set of columns from A_a whose supports[†] are pairwise disjoint and whose columns each have a ones. Let $K_a = |A'_a|$. Discard from A_a the columns of A'_a and any row with a one in A'_a. We are left with a $k_a \times (M - K_a)$ matrix A_{a-1}, where $k_a = N - aK_a$. Clearly, the columns of A_{a-1} have at most $a-1$ ones (indeed, otherwise such a column could be added to the previously discarded set, contradicting its maximality). We continue by doing to A_{a-1} what we did to A_a. That is we define A'_{a-1} to be any maximal set of columns from A_{a-1} whose supports are pairwise disjoint and whose columns each have $a - 1$ ones. Let $K_{a-1} = |A'_{a-1}|$. Then discard from A_{a-1} the columns of A'_{a-1} and any row with a one in A'_{a-1} getting a $k_{a-1} \times (M - K_a - K_{a-1})$ matrix A_{a-2}, where $k_{a-1} = N - aK_a - (a-1)K_{a-1}$.

The process will terminate after at most a steps (when we have a matrix containing only zeros). The union of the columns of the discarded sets form the desired submatrix C with $K = \sum_{i=1}^{a} K_i$. The first step of the algorithm gives $k_a = N - aK_a$, which we rewrite, setting $k_{a+1} = N$, as

$$K_a = \frac{k_{a+1} - k_a}{a}\,.$$

Analogously,

$$K_i = \frac{k_{i+1} - k_i}{i} \qquad \text{for } i = 1, \ldots, a.$$

Now we derive an upper bound for k_i by counting the number of ones in A_{i-1} in two ways: every row of A_{i-1} contains at least v ones, and every column at most $i - 1$ ones, thus

$$vk_i \leq (i-1)(M - K_a - \cdots - K_{i+1}) \leq (i-1)M\,,$$

or equivalently,

$$k_i \leq \frac{(i-1)M}{v}\,.$$

[†] The *support* of a vector is the set of its nonzero coordinates.

So,

$$
\begin{aligned}
K = \sum_{i=1}^{a} K_i &= \sum_{i=1}^{a} \frac{k_{i+1}-k_i}{i} \\
&= \frac{k_{a+1}}{a} + \frac{k_a}{a(a-1)} + \frac{k_{a-1}}{(a-1)(a-2)} + \cdots + \frac{k_2}{2\cdot 1} - k_1 \\
&\le \frac{N}{a} + \frac{M}{v}\left(\frac{1}{a} + \frac{1}{a-1} + \cdots + \frac{1}{2}\right) \le \frac{N}{a} + \frac{M}{v}\ln a\,.
\end{aligned}
$$

The last inequality here follows because $1+1/2+1/3+\cdots+1/n$ is the n-th harmonic number which is known to lie between $\ln n$ and $\ln n+1$. Together with (2.7), this yields $K \le (M/v)(1+\ln a)$, as desired. □

The advantage of this proof is that it can be turned into a simple greedy algorithm which constructs the desired $N \times K$ submatrix A' with column-set C, $|C| = K$:

1. Set $C := \emptyset$ and $A' := A$.
2. While A' has at least one row do:
   - find a column c in A' having a maximum number of ones;
   - delete all rows of A' that contain a 1 in column c;
   - delete column c from A';
   - set $C := C \cup \{c\}$.

2.6.1 Covering designs

An (n,k,l) *covering design* is a family $\mathcal{F}$ of k-subsets of an n-element set (called *blocks*) such that every l-subset is contained in at least one of these blocks. Let $M(n,k,l)$ denote the minimal cardinality of such a design. A simple counting argument (Exercise 1.26) shows that $M(n,k,l) \ge \binom{n}{l}/\binom{k}{l}$.

In 1985, Rödl proved a long-standing conjecture of Erdős and Hanani that for fixed k and l, coverings of size $\binom{n}{l}/\binom{k}{l}(1+o(1))$ exist. Rödl used non-constructive probabilistic arguments. We will now use the Lovász–Stein theorem to show how to *construct* an (n,k,l) covering design with only $\ln\binom{k}{l}$ times more blocks. This is not as sharp as Rödl's celebrated result, but it is constructive. A polynomial-time covering algorithm, achieving Rödl's bound, was found by Kuzjurin (2000).

Theorem 2.17. $M(n,k,l) \le \binom{n}{l}/\binom{k}{l}\left[1+\ln\binom{k}{l}\right]$.

Proof. Let $X = (x_{S,T})$ be an $N \times M$ 0-1 matrix with $N = \binom{n}{l}$ and $M = \binom{n}{k}$. Rows of X are labeled by l-element subsets $S \subseteq [n]$, columns by k-element subsets $T \subseteq [n]$, and $x_{S,T} = 1$ iff $S \subseteq T$. Note that each row contains exactly $v = \binom{n-l}{k-l}$ ones, and each column contains exactly $a = \binom{k}{l}$ ones.

By the Lovász–Stein theorem, there is an $N \times K$ submatrix X' such that X' does not contain an all-0 row and