Sperner's theorem: A maximal number of pairwise incomparable subsets of n-element set does not exceed $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. This bound is achieved.

Proof: Let's prove that if m pair-wise incomparable subsets exist then at the same time there are other m pair-wise incomparble subsets where cardinality of each subset strictly equals $\lfloor \frac{n}{2} \rfloor$. Obviously, it results in Schperner's theorem because the number of various subsets of n-element set including $\lfloor \frac{n}{2} \rfloor$ elements will equal strictly $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

So let's examine pair-wise incomparable subsets $A_1, \ldots, A_m$ (an antichain). We'll choose those subsets which possess the highest cardinality. Let's call them $C_1, \ldots, C_q$. The cardinality of every subset C_i will equal k and k will be more than $\lfloor \frac{n}{2} \rfloor$.

Then every subset C_i will correspond with all those $(k-1)$-element subsets which appear as a result of throwing one element out of the subset. Apparently every C_i will correspond with strictly k $(k-1)$-element subsets. And every $(k-1)$-element subset will correspond with not more than $(n-k+1)$ various subsets C_i. Let's notice that $k \geq (n-k+1)$.

Therefore we get a bigraph (its left part consists of subsets C_i, and its right part consists of corresponding subsets) where the number of edges coming from every left part vertice is equal or more than the number of edges coming into every right part vertice. Let's examine any x vertices from left part and y corresponding with them vertices from right part. There are kx edges coming from the left part and not more than $(n-k+1)y$ edges coming into the right part. Therefore $x \leq y$ ($kx \leq (n-k+1)y \leq ky$). By Hall's marriage theorem there is a matching between all the left part vertices and some right part vertices which we'll call $D_1, \ldots, D_q$.

We'll create a new antichain: take $A_1, \ldots, A_m$, exclude $C_1, \ldots, C_q$ and add $D_1, \ldots, D_q$. All the subsets are still pair-wise incomparable.
After several similar operations we'll get that the number of our subsets won't change but they all will include not more than $\lfloor \frac{n}{2} \rfloor$ elements. The same way we'll get that they will include not less than $\lfloor \frac{n}{2} \rfloor$ elements. QED

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