On Sperner's Theorem

Let $\mathcal{F}$ be a family of subsets of $[n]$ with no chain longer than k. Then

$$|\mathcal{F}| \leq \text{Sum of the largest } k \text{ binomial coefficients in } n.$$

Proof. The claim is trivial for $n = 1$ and all k. Assume the claim is true for ground sets of sizes at most n and take a family $\mathcal{F}$ of subsets of $[n+1]$ having no chain longer than k. We can write $\mathcal{F}$ as a disjoint union $\mathcal{F} = \mathcal{G} \cup \mathcal{H}$, where $\mathcal{G} = \{\text{sets in } \mathcal{F} \text{ containing } n+1\}$ and $\mathcal{H} = \{\text{sets in } \mathcal{F} \text{ not containing } n+1\}$. Consider a subfamily $\mathcal{M} \subset \mathcal{H}$ which consists of sets which are the top sets of some some chain of length k in $\mathcal{H}$ and add $\mathcal{M}$ to $\mathcal{G}$. Write $\mathcal{G}' = \mathcal{G} \cup \mathcal{M}$ and $\mathcal{H}' = \mathcal{H}/\mathcal{M}$. Then $\mathcal{F}$ is a disjoint union of $\mathcal{G}'$ and $\mathcal{H}'$. We have that $\mathcal{H}'$ has no chain longer than $k-1$ as we shortened the maximum length chains. Also, no element in $\mathcal{M}$ is contained in an element of $\mathcal{G}$ as in this way we would have a chain longer than $k+1$ in $\mathcal{F}$. Thus $\mathcal{G}'$ corresponds to a set system on the ground set $[n]$ by removing the element $n+1$ from elements of $\mathcal{G}$. This system has no chain longer than $k+1$ as elements from $\mathcal{M}$ can extend a chain in $\mathcal{G}$ by at most one. Thus, by induction, the size of $\mathcal{G}'$ is no more than the $k+1$ largest binomial coefficients in n and $\mathcal{H}'$ has no more elements than the sum of $k-1$ largest binomial coefficients in n. By Pascal's identity this gives in total the sum of k largest binomial coefficients in $n+1$.