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AN ANALYSIS OF THE GREEDY HEURISTIC FOR INDEPENDENCE SYSTEMS

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The worst case behaviour of the greedy heuristic for independence systems is analyzed by deriving lower bounds for the ratio of the greedy solution value to the optimal value. For two special independence systems, this ratio can be bounded by 1/2, for two other independence systems, it converges with increasing problem size to zero. The main theorem states that for every independence system $(E, \mathcal{F})$ the ratio is bounded by $1/k$, k such that $(E, \mathcal{F})$ can be represented as the intersection of k matroids.

1. Introduction

Since Cook [2] and Karp [8] have shown that many notoriously hard problems in combinatorial optimization are equivalent in the sense that either all or none of them can be solved in polynomial time, the study and analysis of fast heuristic algorithms has become more interesting again.

A great number of these heuristics are variants of the well-known *greedy heuristic* for the problem of finding an independent set with maximum weight of a given independence system.

Consider an independence system $(E, \mathcal{F})$, E being an arbitrary finite set and $\mathcal{F}$ a system of subsets of E with the property:

$$F \subseteq G \in \mathcal{F} \implies F \in \mathcal{F}.$$

The elements of $\mathcal{F}$ are called $\mathcal{F}$-independent sets or simply *independent* sets. Moreover, consider a weight function $c : E \to \mathbf{R}^+$ and the optimization problem

$$\left.\begin{array}{r} c(F) = \text{Max!} \\ F \in \mathcal{F}, \end{array}\right\} \tag{1}$$

where $c(F) = \sum_{e \in F} c(e)$. Let $\{e_1, e_2, \ldots, e_n\}$ be a numbering of E with

$$i \leq j \implies c(e_i) \geq c(e_j).$$

Let $E_i = \{e_1, \ldots, e_i\}$ for $1 \leq i \leq n$. A subset $F \subseteq E$ is called *greedy solution* of (1) if, for $1 \leq i \leq n$, $F \cap E_i$ is a maximal independent subset of E_i. It is easy to see by induction that a greedy solution is just the set F yielded by the following *greedy heuristic*:

```
begin F = ∅;
    for i = 1 to n do
    begin
        if F ∪ {e_i} ∈ 𝓕 then F = F ∪ {e_i};
    end;
end.
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Recall that an independence system $(E, \mathcal{F})$ is called a *matroid* iff, for any subset $S \subseteq E$, all maximal independent subsets of S have the same cardinality. It is well known that, for a matroid $(E, \mathcal{F})$, every greedy solution F is an optimal solution of (1). In this paper, we consider arbitrary independence systems $(E, \mathcal{F})$ which are not necessarily matroids. We want to characterize the quality of greedy solutions by deriving lower bounds for the ratio $c(F_g)/c(F_0)$ where F_g is a greedy solution and F_0 an optimal solution. A first step towards such a characterization is the following basic theorem.

Several researchers have worked on this theorem. It was conveyed to us by Edmonds [5]. To our knowledge it was first conjectured by Nemhauser; Jenkyns attacked the theorem in his Ph.D. thesis [6]. Independently Baumgarten [1] found some other proof.

Because of the great interest the theorem has received we will below give a very short new proof of it.

Let $(E, \mathcal{F})$ be an independence system and $S \subseteq E$ an arbitrary subset; we define:
lower rank of $S = \operatorname{lr}(S) = \min\{|F| : F$ a maximal independent subset of $S\}$
upper rank of $S = \operatorname{ur}(S) = \max\{|F| : F$ a maximal independent subset of $S\}$.

Obviously, $(E, \mathcal{F})$ is a matroid iff $\operatorname{lr}(S) = \operatorname{ur}(S)$ for any $S \subseteq E$. Hence, the so-called *rank quotient*

$$\min_{S \subseteq E} \frac{\operatorname{lr}(S)}{\operatorname{ur}(S)}$$

can be interpreted as a measure of how much $(E, \mathcal{F})$ differs from being a matroid.

Theorem 1.1. *Let $(E, \mathcal{F})$ be an independence system, F_g a greedy solution and F_0 an optimum solution of (1). Then for any weight function c*

$$1 \geq \frac{c(F_g)}{c(F_0)} \geq \min_{S \subseteq E} \frac{\operatorname{lr}(S)}{\operatorname{ur}(S)}.$$

Proof. Setting $c(e_{n+1}) := 0$ one obtains through a suitable summation that

$$c(F_g) = \sum_{i=1}^{n} |F_g \cap E_i| (c(e_i) - c(e_{i+1})) \tag{2}$$

$$c(F_0) = \sum_{i=1}^{n} |F_0 \cap E_i| (c(e_i) - c(e_{i+1})). \tag{3}$$

As $F_0 \cap E_i \subseteq F_0 \in \mathcal{F}$ we have $|F_0 \cap E_i| \leq \operatorname{ur}(E_i)$.

By the definition of a greedy solution, $F_g \cap E_i$ is a maximal independent subset of E_i, hence $|F_g \cap E_i| \geq \mathrm{lr}(E_i)$ and thus:

$$|F_g \cap E_i| \geq |F_0 \cap E_i| \frac{\mathrm{lr}(E_i)}{\mathrm{ur}(E_i)} \geq |F_0 \cap E_i| \min_{S \subseteq E} \frac{\mathrm{lr}(S)}{\mathrm{ur}(S)}. \tag{4}$$

From (2), (3), and (4) it follows that $c(F_g) \geq \min_{S \subseteq E} \mathrm{lr}(S)/\mathrm{ur}(S) \cdot c(F_0)$.

Corollary. *The bound in Theorem 1.1 is sharp in the following sense: For every independence system $(E, \mathcal{F})$, there exists a weight function $c : E \to \mathbf{R}^+$ and a greedy solution F_g with*

$$\frac{c(F_g)}{c(F_0)} = \min_{S \subseteq E} \frac{\mathrm{lr}(S)}{\mathrm{ur}(S)}.$$

Proof. Let

$$\frac{\mathrm{lr}(S_0)}{\mathrm{ur}(S_0)} = \min_{S \subseteq E} \frac{\mathrm{lr}(S)}{\mathrm{ur}(S)}$$

and

$$|F_l| = \mathrm{lr}(S_0)$$

$$|F_u| = \mathrm{ur}(S_0).$$

Let

$$c(e) := \begin{cases} 1, & e \in S_0 \\ 0, & e \notin S_0. \end{cases}$$

Let $(e_1, e_2, \ldots, e_n)$ be a numbering of E such that

$$e_i \in F_l,\ e_j \in S_0 - F_l,\ e_k \in E - S_0 \implies i < j < k.$$

Obviously such a numbering satisfies ($i \leq j \implies c(e_i) \geq c(e_j)$) and F_l is a greedy solution of (1). Hence

$$\frac{c(F_g)}{c(F_0)} \leq \frac{c(F_l)}{c(F_u)} = \frac{|F_l|}{|F_u|} = \min_{S \subseteq E} \frac{\mathrm{lr}(S)}{\mathrm{ur}(S)}.$$

Applying Theorem 1.1, the corollary follows.

Having Theorem 1.1 and its corollary, it is enough to inspect the rank quotient because every sharp bound for the rank quotient is a sharp bound for $c(F_g)/c(F_0)$ also. In Section 2, we calculate the rank quotient for four special independence systems, the independence systems of the matching problem, the symmetrical travelling salesman problem, the stable set problem, and the acyclic subgraph problem. In Section 3, we show that every independence system can be represented as the intersection of some matroids. The following main theorem states that, for an intersection of k matroids, the rank quotient and hence the quotient $c(F_g)/c(F_0)$, too, is bounded below by $1/k$. We conclude the paper with some remarks about the sharpness of this bound.

2. The rank quotient for some special independence systems

Let $G = (V, E)$ be a finite undirected graph without loops or multiple edges. A subset $F \subseteq E$ is called a *matching of* G, iff no two edges in F are adjacent, i.e. have a vertex in common. Let $\mathcal{F}$ be the set of all matchings of G. Obviously, $(E, \mathcal{F})$ is an independence system. For the so defined $\mathcal{F}$, (1) is called the *matching problem.*

Theorem 2.1. *Let $(E, \mathcal{F})$ be the independence system of the matching problem for the graph G. In the trivial case where every connected component of G is a triangle K_3, a path P_2, a single edge K_2, or a single vertex K_1, we have*

$$\min_{S \subseteq E} \frac{\operatorname{lr}(S)}{\operatorname{ur}(S)} = 1.$$

Otherwise

$$\min_{S \subseteq E} \frac{\operatorname{lr}(S)}{\operatorname{ur}(S)} = \frac{1}{2}.$$

Proof. We show first that

$$\min_{S \subseteq E} \frac{\operatorname{lr}(S)}{\operatorname{ur}(S)} \geq \frac{1}{2}. \tag{5}$$

Let $S \subseteq E$ be any subset and F_1, F_2 maximal independent subsets of S. It is enough to show that $|F_1|/|F_2| \geq 1/2$.

Let $e \in F_2 \setminus F_1$. Since $F_1 \cup \{e\} \subseteq S$ and F_1 is maximal, the set $F_1 \cup \{e\}$ is not independent, i.e. is not a matching of G. Hence there exists an edge $\phi(e) \in F_1$ that is adjacent to e. Since F_2 is a matching, $\phi(e) \in F_1 \setminus F_2$. Thus we have defined a mapping $\phi : F_2 \setminus F_1 \to F_1 \setminus F_2$. Obviously, for any edge in $F_1 \setminus F_2$, there are at most two edges in $F_2 \setminus F_1$ adjacent to it. Hence, ϕ maps at most two edges of $F_2 \setminus F_1$ onto one edge of $F_1 \setminus F_2$. It follows:

$$|F_1 \setminus F_2| \geq |\phi(F_2 \setminus F_1)| \geq \tfrac{1}{2}|F_2 \setminus F_1|,$$

hence

$$|F_1| = |F_1 \setminus F_2| + |F_1 \cap F_2| \geq \tfrac{1}{2}|F_2 \setminus F_1| + \tfrac{1}{2}|F_1 \cap F_2| = \tfrac{1}{2}|F_2|,$$

whence $|F_1|/|F_2| \geq 1/2$.

If every connected component of G is isomorphic to K_3, P_2, K_2, or K_1, obviously $\operatorname{lr}(S)/\operatorname{ur}(S) = 1$ for any $S \subseteq E$. Otherwise G contains a subgraph (V', S) isomorphic to P_3. Since clearly $\operatorname{lr}(S) = 1$, $\operatorname{ur}(S) = 2$, we have the upper bound

$$\min_{S \subseteq E} \frac{\operatorname{lr}(S)}{\operatorname{ur}(S)} \leq 1/2$$

which together with the lower bound (5) proves the theorem.

Now let (V, E) be a complete undirected graph. Let $\mathcal{F}$ be the set of all subsets of Hamiltonian cycles in (V, E). Since the Hamiltonian cycles are the feasible solutions of the symmetrical travelling salesman problem (TSP), $(E, \mathcal{F})$ is called the *independence system of the symmetrical TSP*. Obviously, a subset $F \subseteq E$ belongs to $\mathcal{F}$ iff:

(i) every vertex $v \in V$ is incident to at most two edges of F and
(ii) the partial graph (V, F) contains no non-Hamiltonian cycle.

Theorem 2.2. *Let $(E, \mathcal{F})$ be the independence system of the symmetrical TSP for the complete graph (V, E). Then*

$$\frac{1}{2} \leq \min_{S \subseteq E} \frac{\operatorname{lr}(S)}{\operatorname{ur}(S)} \leq \frac{1}{2} + \frac{3}{2|V|}.$$

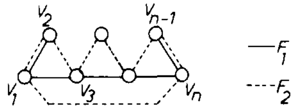
Proof. The proof of the lower bound is analogous to that in the proof of Theorem 2.1. Again we construct a mapping $\phi : F_2 \setminus F_1 \to F_1 \setminus F_2$ which maps as few as possible elements of $F_2 \setminus F_1$ onto the same element of $F_1 \setminus F_2$. Since the complete proof does not contain new ideas but a rather complicated construction, we omit the details.

For the upper bound, we assume $V = \{v_1, v_2, \ldots, v_n\}$ and define the following three sets

$$F_1 = \left\{ v_{2i-1} v_{2i+1} : 1 \leq i \leq \left[\frac{n-1}{2}\right]\right\} \cup \{v_2 v_1, v_n v_{n-1}\},$$

$$F_2 = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1\},$$

$$S = F_1 \cup F_2.$$



Obviously, F_1 and F_2 are maximal independent subsets of S, and

$$\operatorname{lr}(S) \leq |F_1| = \left[\frac{n+3}{2}\right], \qquad \operatorname{ur}(S) = |F_2| = n,$$

hence

$$\min_{S' \subseteq E} \frac{\operatorname{lr}(S')}{\operatorname{ur}(S')} \leq \frac{\operatorname{lr}(S)}{\operatorname{ur}(S)} \leq \frac{1}{2} + \frac{3}{2n}.$$

The next two theorems show that for some independence systems $(E, \mathcal{F})$ the rank quotient converges to 0 as the "problem size" $|E|$ tends to infinity. Consequently, for these independence systems, the greedy heuristic for large problems can be arbitrarily bad.

Let $G = (V, E)$ be a graph. A subset $F \subseteq V$ is called a *stable set* (or independent set, vertex packing) if no two distinct vertices in F are adjacent. Let $\mathcal{F}$ be the system of all stable sets of G. Since then (1) is the *vertex packing problem*, $(V, \mathcal{F})$ is called the independence system of the vertex packing problem in G.

Theorem 2.3. *Let $G = (V, E)$ be a graph containing an induced subgraph isomorphic to the star $K_{1,k}$. Let $(V, \mathcal{F})$ be the independence system of the vertex packing problem in G. Then*

$$\lim_{k\to\infty} \min_{S\subseteq V} \frac{\operatorname{lr}(S)}{\operatorname{ur}(S)} = 0.$$

Proof. Let $S := \{v, v_1, \ldots, v_k\}$ be the set of vertices that induces the subgraph isomorphic to $K_{1,k}$ and let

$$vv_i \in E, \qquad 1 \leq i \leq k, \quad \text{but} \quad v_iv_j \notin E, \quad 1 \leq i, j \leq k.$$

$F_1 = \{v\}$ and $F_2 = \{v_1, \ldots, v_k\}$ are maximal independent subsets of S and

$$\min_{S'\subseteq V} \frac{\operatorname{lr}(S')}{\operatorname{ur}(S')} \leq \frac{\operatorname{lr}(S)}{\operatorname{ur}(S)} = \frac{|F_1|}{|F_2|} = \frac{1}{k} \to 0.$$

Theorem 2.4. *Let $G = (V, E)$ be a complete directed graph, i.e. $E = V \times V$. Let $\mathcal{F}$ be the system of (the edge sets of) all its acyclic subgraphs, i.e.*

$$\mathcal{F} = \{F \subseteq E : F \text{ contains no directed cycle}\}.$$

For the independence system $(E, \mathcal{F})$, we have

$$\lim_{|E|\to\infty} \min_{S\subseteq E} \frac{\operatorname{lr}(S)}{\operatorname{ur}(S)} = 0.$$

Proof. Let $n := |V| = \sqrt{|E|}$, $V = \{v_1, \ldots, v_n\}$ and

$$F_1 = \{v_iv_{i+1} : 1 \leq i \leq n-1\},$$

$$F_2 = \{v_iv_j : 1 \leq j < i \leq n\},$$

$$S = F_1 \cup F_2.$$

Clearly, F_1 and F_2 are maximal independent subsets of S and

$$\min_{S'\subseteq E} \frac{\operatorname{lr}(S')}{\operatorname{ur}(S')} \leq \frac{\operatorname{lr}(S)}{\operatorname{ur}(S)} = \frac{|F_1|}{|F_2|} = \frac{n-1}{\binom{n}{2}} = \frac{2}{\sqrt{|E|}} \to 0.$$

3. The rank quotient for an arbitrary independence system

Having inspected the rank quotient for some special independence systems, we now turn to an arbitrary independence system. We start with the following easy lemma:

Lemma 3.1. *For any independence system $(E, \mathscr{F})$, there exist k matroids $(E, \mathscr{F}^i)$, $1 \leq i \leq k$, with*

$$\mathscr{F} = \bigcap_{i=1}^{k} \mathscr{F}^i.$$

Proof. Let $(E, \mathscr{F})$ be an independence system and $C_1, \ldots, C_k$ the minimal sets in $\{F \subseteq E : F \notin \mathscr{F}\}$, the so-called *circuits* of $(E, \mathscr{F})$. It is easy to see that

$$\mathscr{F} = \bigcap_{i=1}^{k} \mathscr{F}^i \quad \text{where} \quad \mathscr{F}^i := \{F \subseteq E : C_i \not\subseteq F\}.$$

Clearly, any $(E, \mathscr{F}^i)$ is an independence system. Let $S \subseteq E$. If $C_i \not\subseteq S$, S itself is the only maximal independent subset of S. Otherwise, if $C_i \subseteq S$, every maximal independent subset of S consists of all elements of S except one element of C_i. Hence all maximal independent subsets of S have the same cardinality, whence $(E, \mathscr{F}^i)$ is a matroid.

By the proof of Lemma 3.1, the minimum number $k = k(E, \mathscr{F})$, for which an independence system $(E, \mathscr{F})$ can be represented as the intersection of k matroids, is bounded by the number of its circuits. Of course this bound is far from being sharp; it is easy to see (cf. Edmonds [3]) that e.g. the independence system of "bipartite matchings" is an intersection of two matroids and the independence system of the "asymmetric TSP" is an intersection of three matroids. The importance of this minimum number $k(E, \mathscr{F})$ lies in the following main theorem:

Theorem 3.2. *Let $(E, \mathscr{F}^i)$, $1 \leq i \leq k$, be matroids and $\mathscr{F} := \bigcap_{i=1}^{k} \mathscr{F}^i$. For the independence system $(E, \mathscr{F})$ we have*

$$\min_{S \subseteq E} \frac{\operatorname{lr}(S)}{\operatorname{ur}(S)} \geq \frac{1}{k}.$$

Proof. Let $S \subseteq E$ be any subset and F_1, F_2 maximal independent subsets of S. It is enough to show $|F_1|/|F_2| \geq 1/k$.

For $i = 1, \ldots, k$ and $j = 1, 2$, let F_j^i be a maximal $\mathscr{F}^i$-independent subset of $F_1 \cup F_2$ containing F_j. If there were an element $e \in F_2 \setminus F_1$ with $e \in \bigcap_{i=1}^{k} F_1^i \setminus F_1$, then $F_1 \cup \{e\} \subseteq \bigcap_{i=1}^{k} F_1^i \in \mathscr{F}$, a contradiction to the maximality of F_1. Hence each $e \in F_2 \setminus F_1$ can be an element of $F_1^i \setminus F_1$ for at most $k-1$ indices i. It follows

$$\sum_{i=1}^{k} |F_1^i| - k|F_1| = \sum_{i=1}^{k} |F_1^i \setminus F_1| \leq (k-1)|F_2 \setminus F_1| \leq (k-1)|F_2|.$$

By the definition of a matroid, we have

$$|F_1^i| = |F_2^i| \quad \text{for any} \quad i, \ 1 \leq i \leq k.$$

Hence the above inequality implies

$$|F_2| \leq \Big(\sum_{i=1}^{k} |F_2^i| - k\,|F_2|\Big) + |F_2|$$

$$= \sum_{i=1}^{k} |F_1^i| - (k-1)|F_2|$$

$$\leq k\,|F_1|.$$

The theorem follows.

As for the question of sharpness of the bound in Theorem 3.2, we can state the following easy theorem.

Theorem 3.3. *For every integer $k \geq 1$, there exists an independence system $(E, \mathcal{F})$ which is an intersection of k matroids but not an intersection of less than k matroids and which has the property*

$$\min_{S \subseteq E} \frac{\operatorname{lr}(S)}{\operatorname{ur}(S)} = \frac{1}{k}.$$

Proof. Let $G = (V, E)$ be a graph isomorphic to $K_{1,k}$, say

$$V = \{v, v_1, \ldots, v_k\}, \qquad E = \{vv_i : 1 \leq i \leq k\}.$$

By the proof of Theorem 2.3, the independence system $(V, \mathcal{F})$ of the stable sets of G has the property

$$\min_{S \subseteq V} \frac{\operatorname{lr}(S)}{\operatorname{ur}(S)} \leq \frac{1}{k}. \tag{6}$$

Obviously, the k edges $\{v, v_i\}$, $1 \leq i \leq k$, are the circuits of $(V, \mathcal{F})$. Setting $\mathcal{F}^i := \{F \subseteq V : \{v, v_i\} \not\subseteq F\}$, it follows from the proof of Lemma 3.1 that $(V, \mathcal{F})$ is the intersection of the k matroids $\mathcal{F}^i$, $1 \leq i \leq k$. Hence, by Theorem 3.2 and (6),

$$\min_{S \subseteq V} \frac{\operatorname{lr}(S)}{\operatorname{ur}(S)} = \frac{1}{k},$$

and $(V, \mathcal{F})$ cannot be represented as the intersection of less than k matroids.

Theorem 3.3 says that, for every integer k, there *exists* an independence system for which the bound in Theorem 3.2 is sharp. More interesting is the question if the bound is sharp for *every* independence system. The next theorem gives a (negative) answer to this question.

Theorem 3.4. *Let $(E, \mathcal{F})$ be the independence system of the symmetrical TSP for a complete graph (V, E) with $|V| \geq 4$.*

Then there is no integer k such that $(E, \mathcal{F})$ can be represented as the intersection of k matroids and that

$$\min_{S \subseteq E} \frac{\operatorname{lr}(S)}{\operatorname{ur}(S)} = \frac{1}{k}.$$

Proof. Assume there is such an integer k. Then, by Theorem 2.2, $k = 2$ and $(E, \mathcal{F})$ can be represented as the intersection of two matroids $M^i = (E, \mathcal{F}^i)$, $i = 1, 2$.

Let $V = \{v_1, v_2, \ldots, v_n\}$, $n \geq 4$. Then $F := \{v_1v_2, v_2v_3, \ldots, v_nv_1\} \in \mathcal{F}$, but $F \cup \{v_1v_3\} \not\in \mathcal{F}$. Obviously, every "circuit" of $(E, \mathcal{F})$, (i.e. every minimal dependent subset of E) contained in $F \cup \{v_1v_3\}$ is also a circuit of M^1 or of M^2. Now it is well known (cf. e.g. [10]) that, for a matroid, the union of an independent set and a singleton contains at most one circuit. Hence $F \cup \{v_1v_3\}$ contains at most two circuits of $(E, \mathcal{F})$. But in fact, $F \cup \{v_1v_3\}$ contains even four circuits, namely:

$$C_1 = \{v_1v_2, v_1v_3, v_1v_n\}, \qquad C_2 = \{v_3v_1, v_3v_2, v_3v_4\},$$

$$C_3 = \{v_1v_2, v_2v_3, v_3v_1\}, \qquad C_n = \{v_1v_3, v_3v_4, v_4v_5, \ldots, v_nv_1\}.$$

We conclude the paper with some final remarks.

Remark 3.5. Theorem 3.3 and its proof say that there are families of independence systems $(E_n, \mathcal{F}_n)$, $n = 1, 2, \ldots$, with $|E_n| = n$ and the property that the minimum number $k(E_n, \mathcal{F}_n)$ for which $(E_n, \mathcal{F}_n)$ can be represented as the intersection of k matroids is at least a linear function of n. In particular, there is no general constant k^+ with $k(E, \mathcal{F}) \leq k^+$ for all independence systems. We conjecture that there are even families $(E_n, \mathcal{F}_n)$ where $k(E_n, \mathcal{F}_n)$ is a super-linear — perhaps even exponential — function of n. This conjecture cannot be proven by means of Theorem 3.2 because for every independence system $(E, \mathcal{F})$

$$\frac{1}{\min_{S \subseteq E} \frac{\operatorname{lr}(S)}{\operatorname{ur}(S)}} = \max_{S \subseteq E} \frac{\operatorname{ur}(S)}{\operatorname{lr}(S)} \leq |E|.$$

Remark 3.6. Up to now, we inspected the greedy heuristic for the *maximum* problem (1) only. What about a greedy heuristic for a *minimum* problem? Let us consider the problem

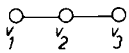
$$\left.\begin{array}{c} c(F) = \text{Min!} \\ F \text{ maximal in } \mathcal{F} \end{array}\right\} \tag{7}$$

where $(E, \mathcal{F})$ is an independence system and $c : E \to \mathbf{R}^+$ is a weight function. The greedy heuristic for (7) starts with the independent set $F = \emptyset$ and, in every step of the algorithm, adds a new element $e \in E \setminus F$ to the current independent set F such that the new set $F \cup \{e\}$ is again independent and that, subject to this condition, e has *minimum* weight c_e.

For the maximum problem (1), it follows from Lemma 3.1, Theorem 3.3, and Theorem 1.1, that for any independence system $(E, \mathcal{F})$ there exists a bound $1/k$ such that

$$\frac{c(F_g)}{c(F_0)} \geqslant \frac{1}{k}$$

for any greedy solution F_g and optimal solution F_0 and *for all weight functions c*. This is not true for the minimum problem (7). E.g. consider the independence system $(V, \mathscr{F})$ of the vertex packing problem in the graph $K_{1,2}$



and the weight function

$$c(v) = \begin{cases} 1, & v = v_1 \\ 2, & v = v_2 \\ M > 2, & v = v_3. \end{cases}$$

The optimal solution of the corresponding minimum problem (7) is $F_0 = \{v_2\}$ and the greedy solution is $F_g = \{v_1, v_3\}$. The quotient $c(F_0)/c(F_g) = 2/M$ converges to 0 as M tends to infinity.

Acknowledgement

After finishing this paper we became aware of a working paper by Jenkyns [7], which coincides with some of our results. We greatfully acknowledge some comments of T.A. Jenkyns on this paper especially for shortening the proof of Theorem 3.2.

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