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Note

Solution of a Problem of A. Ehrenfeucht and J. Mycielski

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A conjecture of A. Ehrenfeucht and J. Mycielski concerning families of subsets is established.

The aim of this note to prove the conjecture posed in [3] by the method used in [1] and [2].

THEOREM. *Let $X = \{1, 2,..., n\}$ be a finite set and $A_1, A_2,..., A_m$, $B_1,..., B_m$ be distinct subsets of X such that*

$$|A_i| = k, \quad |B_i| = l \quad (1 \leqslant i \leqslant m; \quad k, l \text{ fixed}, \quad 1 \leqslant k, l; \quad k + l \leqslant n)$$

and

$$A_i \cap B_j \neq \varnothing \qquad \text{if} \quad i \neq j,$$
$$A_i \cap B_i = \varnothing.$$

Then

$$m \leqslant \binom{k+l}{k}. \tag{1}$$

Proof. 1. Define the subsets C_i, D_i of X in the following way. Let $C_i \cup D_i$ be an arbitrary $(k + l)$-tuple of X $(1 \leqslant i \leqslant \binom{n}{k+l})$, and let C_i consist of the first k elements of this $(k + l)$-tuple, D_i the last l. Denote this system by $\mathscr{F}^i = \{C_i, D_i\}$.

2. Denote the maximal element of C_i by e_i. If $e_i \leqslant e_j$, then every element of C_i is $\leqslant e_i$ and every element of D_j is $> e_j$. Hence $C_i \cap D_j = \varnothing$. Similarly, if $e_i \geqslant e_j$, then $C_j \cap D_i = \varnothing$. We can conclude that either $C_i \cap D_j = \varnothing$ or $C_j \cap D_i = \varnothing$ holds if $i \neq j$.

3. Let $\mathcal{F}_1^i,\dots,\mathcal{F}_{n!}^i$ be the systems formed from $\mathcal{F}^i$ by permuting the elements of X. Their elements are denoted by $\mathcal{F}_u^i = \{C_i^u, D_i^u\}$. From the result of the previous section it follows that either $C_i^u \cap D_j^u = \varnothing$ or $C_j^u \cap D_i^u = \varnothing$ holds ($1 \leqslant u \leqslant n!$).

4. Let us count in two different ways the number of pairs $(\mathcal{F}_u^i, (A_v, B_v))$, where $C_i^u = A_v$, $D_i^u = B_v$. Fix first u. If $C_i^u = A_v$, $D_i^u = B_v$, $C_j^u = A_w$, $D_j^u = B_w$ for some $1 \leqslant v < w \leqslant m$, then $C_i^u \cap D_j^u \neq \varnothing$ and $C_j^u \cap D_i^u \neq \varnothing$ by the suppositions of the theorem, and it contradicts our result in Section 3. It means, to every u we can have at most one (A_v, B_v) with the given property. The number of pairs $(\mathcal{F}_u^i, (A_v, B_v))$ is at most $n!$.

On the other hand, fixing (A_v, B_v), we can choose $\binom{n}{k+l}$ sets (C_i, D_i) to permute into (A_v, B_v). If we fix it, the number of such permutations is $k!\, l!\, (n-k-l)!$ This means that the exact number of $\mathcal{F}_u^i$'s is

$$\binom{n}{k+l} k!\, l!\, (n-k-l)!$$

(not depending on v) and the number of pairs is

$$m \binom{n}{k+l} k!\, l!\, (n-k-l)! \leqslant n!\,.$$

This inequality is equivalent to (1). The proof is completed.

It is easy to see that (1) is the best possible relation, because choosing $|X| = k + l$ and choosing all the k-tuples for C_i ($D_i = X - C_i$), the obtained system satisfies the conditions of the theorem, and the equality in (1).

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Recently I learned, L. Lovász and J. Mycielski also proved this theorem by use of a theorem of Bollobás [4]. They could prove the unicity of the optimal family, too.

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