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AN INEQUALITY ON BINOMIAL COEFFICIENTS

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The purpose of this note is to establish the following result.

Theorem. *For $n \geq 13$,*

$$\sum_{i=0}^{h-1} \binom{n}{i} > \binom{n}{h},$$

if and only if

$$h \geq \left\lfloor \frac{n}{3} \right\rfloor + 2,$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x.

This result is used in [2] to establish a result on edge-coloring of certain hypergraphs. Part of the theorem was known to Erdös [1], who also suggested a line of proof to establish this result for large n. Our proof, while only partly inductive, does establish the result for all $n \geq 13$. For $n \leq 12$, the condition is $h \geq \lceil \frac{n}{3} \rceil + 1$, as can be verified directly.

Proof of the theorem. Let

$$f(n, h) = \sum_{i=0}^{h-1} \binom{n}{i} - \binom{n}{h}.$$

Then, clearly

$$f(n, h) = f(n-1, h-1) + f(n-1, h). \tag{1}$$

It is also easy to show that $f(n, h)$, for a fixed n, begins with $f(n, 0) = -1$, decreases to a minimum value for $h = h^*(n)$, and then increases, eventually becoming positive. To prove this assertion, and to determine $h^*(n)$,

$$\begin{aligned} f(n,h) - f(n,h-1) &= \sum_{i=0}^{h-1} \binom{n}{i} - \binom{n}{h} - \sum_{i=0}^{h-2} \binom{n}{i} + \binom{n}{h-1} \\ &= 2\binom{n}{h-1} - \binom{n}{h} \\ &= \binom{n}{h}\left(\frac{2h}{n-h+1} - 1\right) \\ &= \binom{n}{h}\frac{3h-n-1}{n-h+1}. \end{aligned}$$

Hence, the value for $h^*(n)$ is given by

$$h^*(3m-2) = m-1;$$

$$h^*(3m-1) = m-1 \text{ or } m;$$

$$h^*(3m) = m.$$

For $n = 3m-1$, the maximum value of $f(n,h)$ is achieved for both $h = m$ and $h = m-1$; that is,

$$f(3m-1, m) = f(3m-1, m-1) \geqslant f(3m-1, h). \tag{2}$$

The theorem is equivalent to showing

$$f(3m, m+1) < 0, \tag{3}$$

and

$$f(3m-1, m+1) > 0. \tag{4}$$

To show that (3) and (4) establish the theorem we first see that they imply

$$f(3m, h) < 0, h \leqslant m+1, \tag{5}$$

and

$$f(3m-1, h) > 0, h \geqslant m+1. \tag{6}$$

To show

$$f(3m, h) > 0, h \geqslant m+2, \tag{7}$$

we use (1) to derive

$$f(3m, m+2) = f(3m-1, m+1) + f(3m-1, m+2) > 0,$$

by (6). The result (7) then follows from $h^*(3m) = m$. Next,

$$f(3m-1, h) < 0 \quad \text{if } h \leqslant m, \tag{8}$$

follows from (3) and (1) to derive

$$0 > f(3m, m+1) = f(3m-1, m) + f(3m-1, m+1).$$

By (4), $f(3m-1, m+1) > 0$, so $f(3m-1, m) < 0$. Then, (8) follows from $h^*(3m) = m$. The inequality

$$f(3m-2, h) < 0, \quad h \leqslant m, \tag{9}$$

follows from (2) and (1) which imply

$$f(3m-2, m-2) = f(3m-2, m). \tag{10}$$

Now, $h^*(3m-2) = m-1$ so $f(3m-2, m-2) < 0$. Hence, $f(3m-2, m)$ is also negative and so is every $f(3m-2, h)$, $h \leq m$.

Finally,

$$f(3m-2, h) > 0, \qquad h \geq m+1, \tag{11}$$

follows from $f(3m-1, m+1) > 0$, $f(3m-2, m) < 0$, and (1) in the same way (8) was proven.

Therefore, it suffices to prove (3) and (4). The result (3) can be proven directly or by induction. We give the shorter, direct proof. An equivalent form of (3) is

$$\left(\sum_{i=0}^{m} \binom{3m}{i}\right) \Big/ \binom{3m}{m+1} < 1,$$

or

$$\frac{m+1}{2m} + \frac{(m+1)m}{2m(2m+1)} + \frac{(m+1)m(m-1)}{2m(2m+1)(2m+2)} + \cdots + \frac{(m+1)!}{2m \cdots 3m} < 1.$$

Using $(m-i)/(2m+1+i) < \frac{1}{2}$ for $i \geq 2$, it suffices to show

$$\frac{m+1}{2m} + \frac{(m+1)m}{2m(2m+1)} + \frac{(m+1)m(m-1)}{2m(2m+1)(2m+2)} + 2\frac{(m+1)m(m-1)(m-2)}{2m(2m+1)(2m+2)(2m+3)} < 1,$$

or

$$\frac{16m^3 + 29m^2 + 29m + 6}{16m^3 + 32m^2 + 12m} < 1,$$

which holds for $m \geq 7$. For $m = 5$ and 6, (3) holds as can be verified by computing the expressions. We remark that (3) does not hold for $m \leq 4$; those cases being the only exception to the theorem for $n \leq 12$.

It, thus, remains to show (4). Let $g(h) = f(3h-1, h+1)$. Then,

$$\begin{aligned}
g(h+1) &= \sum_{i=0}^{h+1} \binom{3h+2}{i} - \binom{3h+2}{h+2} \\
&= 2\sum_{i=0}^{h+1} \binom{3h+1}{i} - \binom{3h+1}{h+1} - \binom{3h+2}{h+2} \\
&= 4\sum_{i=0}^{h+1} \binom{3h}{i} - 2\binom{3h}{h+1} - \binom{3h+1}{h+1} - \binom{3h+2}{h+2} \\
&= 8\sum_{i=0}^{h} \binom{3h-1}{i} + 4\binom{3h-1}{h+1} - 2\binom{3h}{h+1} - \binom{3h+1}{h+1} - \binom{3h+2}{h+2} \\
&= 8g(h) + 12\binom{3h-1}{h+1} - 2\binom{3h}{h+1} - \binom{3h+1}{h+1} - \binom{3h+2}{h+2} \\
&= 8g(h) + \binom{3h}{h}\frac{12(2h)(2h-1)}{3h(h+1)} - 2\frac{2h}{h+1} - \frac{3h+1}{h+1} - \frac{(3h+2)(3h+1)}{(h+2)(h+1)} \\
&= 8g(h) - \frac{20}{(h+2)(h+1)}\binom{3h}{h}.
\end{aligned}$$

Noting that $g(1)=f(2,2)=2$, we obtain

$$g(h+1)=8^h\cdot 2-20\sum_{i=1}^{h}\frac{\binom{3i}{i}8^{h-i}}{(i+2)(i+1)}$$
$$=8^h\left(2-20\sum_{i=1}^{h}\frac{\binom{3i}{i}8^{-i}}{(i+2)(i+1)}\right).$$

In order to show $g(h)\geqslant 0$, it suffices to establish

$$2-20\sum_{i=1}^{\infty}\frac{\binom{3i}{i}8^{-i}}{(i+2)(i+1)}\geqslant 0.$$

In fact, equality holds, as will be shown; that is,

$$\sum_{i=1}^{\infty}\frac{\binom{3i}{i}8^{-i}}{(i+2)(i+1)}=\frac{1}{10}.$$

Let

$$F(x)=\sum_{i=1}^{\infty}\binom{3i}{i}x^i(1-x)^{2i}$$
$$=\sum_{i=0}^{\infty}\binom{3i}{i}x^i\sum_{j=0}^{2i}\binom{2i}{j}x^j(-1)^j-1$$
$$=\sum_{i=0}^{\infty}\binom{3i}{i}\sum_{n=i}^{3i}\binom{2i}{n-i}x^n(-1)^{n-i}-1,\quad\text{where } i+j=n,$$
$$=\sum_{n=0}^{\infty}\left[\sum_{i=n/3}^{n}\binom{3i}{i}\binom{2i}{n-i}(-1)^{n-i}\right]x^n-1$$
$$=\sum_{n=0}^{\infty}3^nx^n-1,$$

using

$$3^n=\sum_{i=\lceil n/3\rceil}^{n}\binom{3i}{n}\binom{n}{i}(-1)^{n+i}=\sum_{i=\lceil n/3\rceil}^{n}\binom{3i}{i}\binom{2i}{n-i}(-1)^{n-i}$$

(see [3, p. 51], for the first of the above equalities). Here $\lceil x\rceil$ means the smallest integer greater than or equal to x. Hence,

$$F(x)=\frac{1}{1-3x}-1=\frac{3x}{1-3x}.$$

Now,

$$\sum_{i=1}^{\infty}\frac{\binom{3i}{i}8^{-i}}{(i+2)(i+1)}=8\int_0^{1/8}\sum_{i=1}^{\infty}\binom{3i}{i}t^i(1-8t)\,dt.$$

$$= 8\int_0^{(3-\sqrt{5})/4} \sum_{i=1}^{\infty} \binom{3i}{i} x^i (1-x)^{2i} (1 - 8x(1-x)^2)(1 - 4x + 3x^2)\,dx,$$

where $t = x(1-x)^2$,

$$= 8\int_0^{(3-\sqrt{5})/4} F(x)(1 - 8x(1-x)^2)(1 - 4x + 3x^2)\,dx$$

$$= 8\int_0^{(3-\sqrt{5})/4} \frac{3x}{1-3x}(1 - 8x(1-x)^2)(1-3x)(1-x)\,dx$$

$$= 8\int_0^{(3-\sqrt{5})/4} 3x(1-x)(1 - 8x(1-x)^2)\,dx$$

$$= \frac{1}{10}.$$

References

[1] P. Erdös, Discussions during the symposium. Algorithmic aspects of combinatorics, Qualium Beach, Vancouver Island, B.C. (May 1976).
[2] E. L. Johnson, On the edge-coloring property for the closure of the complete hypergraphs, Ann. Discrete Math. 2 (1978) 161–171.
[3] J. Riordan, Combinatorial Identities (John Wiley, NY, 1968).