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Bottleneck Extrema

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ABSTRACT

Let E be a finite set. Call a family of mutually noncomparable subsets of E a clutter on E. It is shown that for any clutter $\mathscr{R}$ on E, there exists a unique clutter $\mathscr{S}$ on E such that, for any function f from E to real numbers,

$$\min_{R\in\mathscr{R}} \max_{x\in R} f(x) = \max_{S\in\mathscr{S}} \min_{x\in S} f(x).$$

Specifically, $\mathscr{S}$ consists of the minimal subsets of E that have non-empty intersection with every member of $\mathscr{R}$. The pair $(\mathscr{R}, \mathscr{S})$ is called a blocking system on E. An algorithm is described and several examples of blockings systems are discussed.

1. INTRODUCTION

Gross [7] has described an algorithm and a duality theorem for the bottleneck assignment problem: Given a square array of numbers, find a circling of entries with exactly one circle in each row and one circle in each column so as to maximize the value of the smallest circled entry. (For an interpretation, think of rows of the array as corresponding to men, columns to jobs, on a serial production line, with the entry in row i and column j being the rate at which man i can process items if he is assigned to job j.) An earlier, less efficient algorithm for this problem was given by Fulkerson, Glicksberg, and Gross [5]. The duality theorem proved by Gross is: Let $I = \{1, 2,..., n\}$; let Π be the set of permutations of I; let $|C|$ denote cardinality of C, and let a_{ij} (for $i, j \in I$) be real numbers. Then

$$\max_{\pi\in\Pi} \min_{i\in I} a_{i,\pi(i)} = \min_{\substack{A,B\subset I\\ |A|+|B|=n+1}} \max_{\substack{i\in A\\ j\in B}} a_{ij} .$$

Similarly, the following bottleneck path problem has been considered by Pollack [12], Hu [9], and Fulkerson [4]. Let G be a network (graph) whose arcs (edges) have numerical "weights." Let a and b be two nodes (vertices) in G. Find in G a path P from a to b such that the minimum single arc-weight in P is a maximum. (For an interpretation, think of G as a flow-network with source a, sink b, where the weight of an arc is its flow-capacity.) The duality theorem noted in [4] for bottleneck paths is: The maximum of the minimum weight of an arc in a path from a to b is equal to the minimum of the maximum weight of an arc in a cut separating b from a. Here a *cut* separating b from a is a minimal set of arcs such that deleting them from G leaves a network which contains no path from a to b; "minimal" means that no proper subset has the same property. (If arcs in G are directed, "path" is interpreted to mean "uniformly directed path.")

The well-known traveling salesman problem is to find, in a given graph G whose arcs (possibly directed) have numerical weights, a minimum weight closed path that contains each node of G just once. A closed path that contains each note of G once is called a Hamilton *tour*. The bottleneck traveling salesman problem is to find a Hamilton tour such that the largest arc-weight in the tour is minimum. Gilmore and Gomory [6] have solved a special case of the traveling salesman problem and also a special case of the bottleneck traveling salesman problem.

The reader should now be able to pose bottleneck problems galore. For the moment, we give two more examples. In an undirected graph G whose arcs have weights, find a spanning tree T such that the maximum weight of an arc not in T is minimum. In an undirected graph G whose nodes have weights, find a set C of nodes such that C meets all of the arcs, and such that the maximum node-weight in C is minimum.

2. The Bottleneck Theorem and Threshold Method

Let E be a finite set. A *family* $\mathscr{F}$ *on* E is a family of subsets of E. E is called the *domain* of $\mathscr{F}$ (regardless of whether the union of members of $\mathscr{F}$ is E). We define a *clutter* $\mathscr{R}$ *on* E to be a family $\mathscr{R}$ on E such that no member of $\mathscr{R}$ is contained in another member of $\mathscr{R}$.

The interest cited in bottleneck problems prompts the following theorem.

THEOREM: *For any clutter $\mathscr{R}$ on a finite set E, there exists a unique clutter $\mathscr{S} = b(\mathscr{R})$ on E such that, for any function f from E to real numbers,*

$$\min_{R\in\mathscr{R}} \max_{x\in R} f(x) = \max_{S\in\mathscr{S}} \min_{x\in S} f(x). \tag{1}$$

Specifically, $\mathscr{S} = b(\mathscr{R})$ is the clutter consisting of the minimal subsets of E that have non-empty intersection with every member of $\mathscr{R}$.

COROLLARY. $b(b(\mathscr{R})) = \mathscr{R}$.

We call $\mathscr{S}$ the *blocking clutter*, or simply the *blocker*, of $\mathscr{R}$. By a *blocking system* on E we shall mean any two families $\mathscr{R}$ and $\mathscr{S}$ on E that satisfy (1) for every f, regardless of whether $\mathscr{R}$ and $\mathscr{S}$ are clutters.

If $\mathscr{F}$ is any family on E, in place of $\mathscr{R}$ and $\mathscr{S}$, respectively, in (1), denote the left side of (1) as $u(\mathscr{F},f)$ and the right side of (1) as $w(\mathscr{F},f)$. The bottleneck problems, determine $u(\mathscr{F},f)$ and determine $w(\mathscr{F},f)$, where $\mathscr{F}$ is any family on E, reduce to the case in which $\mathscr{F}$ is a clutter on E, since clearly:

(2) *Where f is any real-valued function on E, $u(\mathscr{F},f) = u(\mathscr{R},f)$ and $w(\mathscr{F},f) = w(\mathscr{R},f)$ for any families $\mathscr{F}$ and $\mathscr{R}$ on E such that every member of $\mathscr{F}$ has some member of $\mathscr{R}$ as a subset and such that $\mathscr{R} \subset \mathscr{F}$.*

In particular, these equations hold if $\mathscr{F}$ is arbitrary and $\mathscr{R}$ consists of those members of $\mathscr{F}$ that contain no other member of $\mathscr{F}$.

Central to our subject is the following property for a pair $(\mathscr{R}, \mathscr{S})$ of families on E:

(3) *For any partition of E into two sets E_0 and E_1 ($E_0 \cap E_1 = \phi$ and $E_0 \cup E_1 = E$), either a member of $\mathscr{R}$ is contained in E_0 or a member of $\mathscr{S}$ is contained in E_1, but not both.*

The bottleneck theorem, above, follows immediately from Lemmas A, B, and C.

LEMMA A. *Any blocking system satisfies property* (3).

LEMMA B. *For any clutter $\mathscr{R}$ on a set E, the $\mathscr{S} = b(\mathscr{R})$ specified in the theorem is the one and only clutter on E such that* (3) *holds.*

LEMMA C. *Any pair $(\mathscr{R}, \mathscr{S})$ of families on E satisfying* (3) *is a blocking system.*

The proof of Lemma C will be an algorithm, based on (3), for computing $u(\mathscr{R},f)$ and $w(\mathscr{S},f)$, thereby showing them to be equal. This algorithm, which we call the threshold method, requires only a small number of "steps," where each step consists mainly of deciding, for a given bipartition (E_0, E_1) of E, which of the two alternatives in (3) holds. Thus the threshold method is a good algorithm provided that there is a good algorithm for the latter.

PROOF OF LEMMA A. That a blocking system satisfies (3) follows from equation (1), where $f(x) = 0$ for $x \in E_0$ and $f(x) = 1$ for $x \in E_1$. If the resulting value of $u(\mathscr{R}, f) = w(\mathscr{S}, f)$ is 0, then some member of $\mathscr{R}$ is contained in E_0 and no member of $\mathscr{S}$ is contained in E_1. If the resulting value of $u(\mathscr{R}, f) = w(\mathscr{S}, f)$ is 1, then no member of $\mathscr{R}$ is contained in E_0 and some member of $\mathscr{S}$ is contained in E_1.

PROOF OF LEMMA B. It is convenient to consider another operator, $d(\mathscr{R})$, defined for every clutter $\mathscr{R}$ on E: $d(\mathscr{R})$ consists of the complements in E of the members of $\mathscr{R}$. Clearly $d(\mathscr{R})$ is a clutter on E, and $d(d(\mathscr{R})) = \mathscr{R}$. Property (3) seems more transparent in terms of $\mathscr{R}$ and the family $p(\mathscr{R}) = d(b(\mathscr{R}))$, and so it is useful to view $b(\mathscr{R})$ as $d(p(\mathscr{R}))$.

For any clutter $\mathscr{R}$ on E, define $p(\mathscr{R})$ to consist of the maximal subsets of E that contain no member of $\mathscr{R}$. Clearly $p(\mathscr{R})$ is a clutter on E. Clearly $d(p(\mathscr{R}))$ is the $\mathscr{S} = b(\mathscr{R})$ specified in the theorem.

Property (3) for clutters $\mathscr{R}$ and $\mathscr{S} = d(p(\mathscr{R}))$ is equivalent to the obvious fact that:

(4) *Every subset E_0 of E either contains a member of $\mathscr{R}$ or is contained in a member of $p(\mathscr{R}) = d(\mathscr{S})$, but not both.*

The equivalence follows because E_0 is contained in a member of $d(\mathscr{S})$ if and only if $E_1 = E - E_0$ contains a member of $\mathscr{S}$.

We must verify that $\mathscr{S} = d(p(\mathscr{R}))$ is the only clutter on E such that (3) holds for $(\mathscr{R}, \mathscr{S})$. This follows because $p(\mathscr{R})$, as defined, is the only clutter on E for which (4) holds. To see this, suppose that clutter $\mathscr{P}$, in place of $p(\mathscr{R})$, satisfies (4). For any $P \in \mathscr{P}$, P cannot contain a member of $\mathscr{R}$ since P is contained in a member of $\mathscr{P}$ (itself). Because $\mathscr{P}$ is a clutter, any set $A \subset E$ which properly contains P is not contained in any member of $\mathscr{P}$. Therefore, by (4), any such A contains a member of $\mathscr{R}$. Thus P is a maximal subset of E containing no member of $\mathscr{R}$, and thus we conclude that $\mathscr{P} \subset p(\mathscr{R})$. On the other hand, for any $Q \in p(\mathscr{R})$, Q is not properly contained in any member of $\mathscr{P}$ since $p(\mathscr{R})$ is a clutter and since $\mathscr{P} \subset p(\mathscr{R})$. Therefore we have $Q \in \mathscr{P}$, since otherwise $E_0 = Q$ would be a set which contains no member of $\mathscr{R}$ and which is contained in no member of $\mathscr{P}$. Thus we conclude that $\mathscr{P} = p(\mathscr{R})$. This completes the proof of Lemma B.

PROOF OF LEMMA C. Suppose that $(\mathscr{R}, \mathscr{S})$ is any pair of families on E that satisfies property (3), and let f be any real-valued function on E. We shall show that equation (1) holds, i.e., that $(\mathscr{R}, \mathscr{S})$ is a blocking system.

To compute $u(\mathscr{R}, f)$, we use the following "threshold method." It is different from previously proposed algorithms for special bottleneck problems.

Choose elements $x \in E$ one after another in order of non-decreasing magnitude of $f(x)$ until the set of chosen elements first contains an $R \in \mathscr{R}$. When this happens, stop. Denote the final set of chosen elements by B_u, denote the last chosen element by x_u, and denote any one of the members of $\mathscr{R}$ contained in B_u by R_u (there may be several). We have $x_u \in R_u$ since $B_u - x_u$ contains no $R \in \mathscr{R}$. Element x_u maximizes f over B_u and thus over R_u. Therefore $u(\mathscr{R}, f) \leqslant f(x_u)$. Since $B_u - x_u$ contains every x such that $f(x) < f(x_u)$, if there were an $R \in \mathscr{R}$ such that

$$\max_{x \in R} f(x) < f(x_u),$$

we would have $R \subset B_u - x_u$. Therefore $u(\mathscr{R}, f) = f(x_u)$.

By property (3), $B_w = E - (B_u - x_u)$ contains a member S_w of $\mathscr{S}$. By property (3), $B_w - x_u = E - B_u$ contains no member of $\mathscr{S}$, and so we have $x_u \in S_w$. Element x_u minimizes f over B_w and thus over S_w. Therefore $f(x_u) \leqslant w(\mathscr{S}, f)$. Since $B_w - x_u$ contains every x such that $f(x_u) < f(x)$, if there were an $S \in \mathscr{S}$ such that

$$f(x_u) < \min_{x \in S} f(x),$$

we would have $S \subset B_w - x_u$. Therefore $f(x_u) = w(\mathscr{S}, f)$. This completes the proof of Lemma C and the bottleneck theorem.

One can of course use the "dual threshold method" instead. That is, choose elements $x \in E$ one after another in order of non-increasing magnitude of $f(x)$ until the set of chosen elements first contains an $S \in \mathscr{S}$.

The concept of a blocking system of clutters $\mathscr{R}$ and $\mathscr{S}$ or of the blocking system of families R^+ and S^+, where $\mathscr{R}^+$ and $\mathscr{S}^+$ consist of all supersets of members of $\mathscr{R}$ and $\mathscr{S}$, respectively, arises in other contexts besides bottleneck extrema (see [8], [10], [11], [13], for example). In particular, the families $\mathscr{R}^+$ and $\mathscr{S}^+$ are Boolean duals of each other (the Boolean dual $\mathscr{F}^*$ of a family $\mathscr{F}$ on E consists of those subsets $H \subset E$ such that $E - H$ is not a member of $\mathscr{F}$).

3. Some Examples of Blocking Systems

A *transversal* of an n by n array M is a subset of the positions in M such that there is exactly one member of the subset in each line of M. (A *line* of an array is either a row or a column of the array.) If clutter $\mathscr{S}$ consists of the transversals of M, its blocker $\mathscr{R}$ consists of the h by k subarrays of M with $h + k = n + 1$. This is the blocking system for the bottleneck assignment problem.

As stated earlier, if $\mathscr{S}$ consists of the arc-sets of paths from node a to node b in a graph G (perhaps directed), then the members of its blocker $\mathscr{R}$ are called the cuts separating b from a.

If clutter $\mathscr{R}$ consists of the arc-sets that are complementary to spanning trees in a graph G, then $\mathscr{S}$ consists of the arc-sets of circuits (polygons) in G.

If $\mathscr{R}$ consists of the minimal sets of nodes that meet all arcs in a graph G, then $\mathscr{S}$ consists of the pairs of adjacent nodes in G.

In each of these examples, there is a good algorithm for recognizing whether a given subset E_0 of the domain E contains a member of the clutter $\mathscr{R}$ or whether its complement $E_1 = E - E_0$ contains a member of clutter $\mathscr{S}$.

Very often it is difficult to find a useful description of the blocking clutter of a simply described clutter, and very often it is difficult to evaluate a bottleneck extremum. In view of the threshold method for bottleneck extrema, it is clear that having a good algorithm for a bottleneck problem, defined by any clutter $\mathscr{R}$ of some class of clutters and by any function f on the domain E of $\mathscr{R}$, is equivalent to having a good algorithm for determining, for any $\mathscr{R}$ of the class and any subset $E_0 \subset E$, whether or not E_0 contains a member of $\mathscr{R}$, i.e., for determining whether E_0 contains a member of $\mathscr{R}$ or whether $E - E_0$ contains a member of $b(\mathscr{R})$. A necessary, though not sufficient, condition for the latter is having a good algorithm for recognizing whether any given subset of E is itself a member of $\mathscr{R}$ or a member of $b(\mathscr{R})$. For any clutter $\mathscr{R}$ of direct interest, it is likely that its members are easily recognizable. Unfortunately, this does not imply that the same is true for $b(\mathscr{R})$.

The theorems below may be interpreted as describing good algorithms for recognizing members of the blocking clutters of certain clutters $\mathscr{R}$. Good algorithms are known also (though we will not describe them here) for determining, for any one of these particular clutters and any subset of its domain, whether or not the subset contains a member of the clutter.

The description of the blocking clutter of the clutter of transversals in a square array is, in spite of its simple appearance, a quite substantial theorem. In view of property (3), it is clearly equivalent to the following: For any n by n array M and any subset E_0 of positions in M, E_0 contains no transversal of M if and only if there are $2n - (n + 1) = n - 1$ lines of the array that together contain all of E_0. This is a special case of the well-known König theorem: For any rectangular array M and any subset E_0 of its positions, the maximum cardinality of a matching contained in E_0 equals the minimum number of lines that together contain all of E_0. (A *matching* is a set of positions, no two of which lie in the same line.) The König theorem is equivalent to the following description of a more general class of blocking systems: If clutter $\mathscr{R}$ consists of the matchings of cardinality t in an m by n array, then $b(\mathscr{R})$ consists of all h by k subarrays such that $h + k = m + n - t + 1$. A good algorithm for determining whether a given subset of the positions in an m by n array contains a matching of size t is described in [3].

Another blocking system based on m by n arrays can be obtained from the linear programming transportation problem: Let $X = (x_{ij})$ be an extreme solution (basic feasible solution) of the constraints

$$\sum_{j=1}^{n} x_{ij} = r_i \,, i = 1,..., m, \qquad \sum_{i=1}^{m} x_{ij} = s_j \,, j = 1,..., n, \qquad x_{ij} \geqslant 0,$$

where r_i and s_j are given non-negative numbers satisfying

$$\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j \,.$$

The *support* of X is the subset of positions (i, j) such that $x_{ij} > 0$. Then the family of all supports of extreme solutions X is a clutter $\mathscr{R}$ on the domain E of positions (i, j), and $b(\mathscr{R})$ consists of all minimal subarrays $I \times J$ (where $I \subset \{1, 2,..., m\}$, $J \subset \{1, 2,..., n\}$) such that

$$\sum_{i \in I} r_i + \sum_{j \in J} s_j > \sum_{i=1}^{m} r_i \,.$$

This description of $b(\mathscr{R})$ can be deduced from the max-flow min-cut theorem of Ford and Fulkerson [3]. Here the bottleneck problem, evaluate $u(\mathscr{R}, f)$ for any given real-valued function f on the set E of positions (i, j), has the interpretation: Satisfy all the "demands" s_j from the "supplies" r_i in the least time, f_{ij} being the transportation time from supply point i to demand point j. There are good network-flow algorithms for determining whether a given subset of positions contains the support of a solution X.

Let E consist of all the unordered pairs of objects in a finite set V. A *perfect matching* of V is a subset of E whose members are disjoint and together contain all of V. Let clutter $\mathscr{R}$ consist of all the perfect matchings of V. Then $\mathscr{S} = b(\mathscr{R})$ consists of the subsets $S(\mathscr{P})$ of E obtained as follows: $\mathscr{P}$ is any family of mutually disjoint, odd-cardinality subsets of V such that $| V | - | \cup(\mathscr{P})| = | \mathscr{P} | - 2$; $x \in E$ is a member of $S(\mathscr{P})$ if and only if the two members of x are members of different members of $\mathscr{P}$. This result is equivalent to Tutte's theorem characterizing those subsets E_0 of E that contain a perfect matching [14]. (See [1].) Edmonds [2] has given a good algorithm for determining whether a subset E_0 of E contains a perfect matching.

One of the many classes of clutters $\mathscr{R}$ for which $b(\mathscr{R})$ is generally a mystery is where $\mathscr{R}$ consists of the arc-sets of Hamilton tours in a graph. The bottleneck traveling salesman problem, like the traveling salesman problem, is also a mystery. There is no known good algorithm for determining whether a given subset of the arcs of a graph contains a member of $\mathscr{R}$.

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