ON GENERALIZED GRAPHS

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A *generalized graph* consists of a set of n vertices and a collection of k-tuples of these vertices (cf. TURÁN [1]). In what follows we shall refer to such a configuration as an *edge-graph* if $k=2$ and, usually, simply as a graph if $k>2$. A *complete m-graph* has m vertices and $\binom{m}{k}$ k-tuples. We say that a graph G is *m-saturated* if it contains no complete m-graph but loses this property when any new k-tuple is added.

TURÁN [2] proved the following theorem on edge-graphs in 1941: Let $n = g(m-1)+r$, where g, m, and r are integers such that $g \geqq 1, m \geqq 3, 0 \leqq r \leqq m-1$, and $n \geqq m$. Then an m-saturated edge-graph of n vertices can have at most

$$E_m = \frac{m-2}{2(m-1)}(n^2-r^2)+\binom{r}{2}$$

edges. The dual problem was recently solved by ERDŐS, HAJNAL, and MOON [3] who showed that such an edge-graph must have at least

$$e_m = (m-2)(n-m+2)+\binom{m-2}{2} = \frac{m-2}{2}(2n-m+1)$$

edges. These two results can be combined as follows: If G is an m-saturated edge-graph of n vertices and e edges, then

$$e_m \leqq e \leqq E_m.$$

The extremal edge-graphs for which $e=e_m$ or $e=E_m$ are also characterized in these papers.

Corresponding problems can be stated for generalized graphs. Let G be a $(k+l)$-saturated graph with n vertices and t k-tuples, where it is understood that $k+l \leqq n$. Let the maximum and minimum values of t over the class of such graphs G be denoted by $T_{k,l}$ and $t_{k,l}$, respectively. The value of $T_{k,l}$ is not known when $k \geqq 3$ and $l \geqq 1$. (Some rather rough estimates are contained in [4].) The object in this paper is to determine $t_{k,l}$ for all k and l.

Let $M(n, k, l)$ be the generalized graph of n vertices defined as follows: If the vertices have been numbered $1, 2, \ldots, n$ then the k-tuples are all those which contain at least one of the first l vertices. $M(n, k, l)$ is obviously $(k+l)$-saturated, and it has

$$t(n, k, l) = \binom{n}{k} - \binom{n-l}{k}$$

k-tuples.

We now prove the following theorem.

THEOREM 1. $$t_{k,l} = t(n, k, l),$$

and $M(n,k,l)$ is the only $(k+l)$-saturated graph with n vertices and $t(n,k,l)$ k-tuples.

Our proof uses the following lemma.

LEMMA. *Let I denote an index set. For every $i \in I$ A_i and B_i are subsets of a set P with p elements satisfying the following conditions:*

1. $A_i \cap B_i = \emptyset$.
2. $A_i \not\subset A_j \cup B_j$, if $i \neq j$.

If there are a_i and b_i elements in A_i and B_i, respectively, then

$$(*) \qquad \sum_{i \in I} \frac{1}{\binom{p-b_i}{a_i}} \leqq 1,$$

with equality if and only if $B_i = B$ for all $i \in I$ and the sets A_i are the q-tuples of the set $P - B$ for some value of q.

REMARK. It follows from the second condition that no subset A_i can be the null set; B_i, however, may be the null set.

PROOF OF THE LEMMA. We use induction on p. If $p = 1$ there can be only one pair of sets A_i and B_i; for these sets $a_i = 1$ and $b_i = 0$, so the inequality holds.

Suppose the inequality holds for all sets P with fewer than p elements and consider a set P with p elements. If there is an index $i_0 \in I$ such that $a_{i_0} + b_{i_0} = p$, then $A_{i_0} \cup B_{i_0} = P$ by condition 1; hence $I = \{i_0\}$, by condition 2, and so

$$\sum_{i \in I} \frac{1}{\binom{p-b_i}{a_i}} = \frac{1}{\binom{p-b_{i_0}}{a_{i_0}}} = 1.$$

Therefore, we may suppose that $a_i + b_i < p$ for all i. Let $P_1, P_2, \ldots, P_p$ denote the $(p-1)$-element subsets of P. For any integer ν between 1 and p let

$$I_\nu = \{i : i \in I \text{ and } A_i \subset P_\nu\}$$

and

$$B_i^{(\nu)} = B_i \cap P_\nu .$$

The number of elements in $B_i^{(\nu)}$ will be denoted by $b_i^{(\nu)}$.

The sets A_i and $B_i^{(\nu)}$ for $i \in I_\nu$ satisfy the conditions of the lemma with respect to the set P_ν; from the induction hypothesis it follows, therefore, that

$$\sum_{i \in I_\nu} \frac{1}{\binom{p-1-b_i^{(\nu)}}{a_i}} \leqq 1. \tag{1}$$

Consequently,

$$\sum_{\nu=1}^{p} \sum_{i \in I_\nu} \frac{1}{\binom{p-1-b_i^{(\nu)}}{a_i}} \leqq p. \tag{2}$$

To express the left-hand side in another way we determine the total contribution of terms associated with an arbitrary value of the index i.

There are $p-a_i-b_i$ sets P_ν containing $A_i \cup B_i$. For each of these value of ν there is a contribution of $\binom{p-1-b_i}{a_i}$ to the above sum. There are b_i sets P_ν which contain A_i but not B_i. For each of these values of ν there is a contribution of $\binom{p-b_i}{a_i}$. There are a_i sets P_ν not containing A_i but these will give no contribution to the sum (2), since i does not belong to the corresponding sets I_ν.

Therefore, the contribution of terms associated with the index i is

$$\frac{(p-a_i-b_i)}{\binom{p-1-b_i}{a_i}} + \frac{b_i}{\binom{p-b_i}{a_i}} = \frac{p}{\binom{p-b_i}{a_i}}.$$

Hence, inequality (2) is equivalent to the inequality

$$\sum_{i \in I} \frac{p}{\binom{p-b_i}{a_i}} \leqq p.$$

But this implies that $(\ast)$ holds, as was to be shown.

We now consider the cases for which equality holds in $(\ast)$. From the preceding argument it follows that this happens if and only if equality holds for each value of ν in (1), i. e. if and only if $(\ast\ast)$ the sets A_i and B_i^ν, for $i \in I_\nu$, have the property stated at the conclusion of the lemma for each value of ν.

We shall show that for arbitrary i and j the sets A_i and A_j have the same number of elements and that $B_i = B_j$.

If $A_i \cup A_j = R \neq P$, then there is a set $P_{\nu_0} \supset R$. From this it follows, appealing to $(\ast\ast)$, that A_i and A_j have the same number of elements and $B_i \cap R = B_j \cap R = \emptyset$. Now we prove $B_i = B_j$. Suppose this does not hold. Then we may suppose that for an element x $x \in B_i, x \notin B_j$. Let y and z be elements of P such that $y \in A_i$, $y \notin A_j$; $z \notin A_i \cup B_i$, $z \in A_j$ (such elements exist because $A_i \not\subset A_j$, and $A_j \not\subset A_i \cup B_i$). The set $P_{\nu_1} = P-y$ contains A_j so by $(\ast\ast)$ there is an index k such that $A_k = A_j - z + x$ and so $x \notin B_k$. But then the set $P_{\nu_2} = P-z$ contains A_i, A_k and B_i, hence by $(\ast\ast)$ $B_k \cap P_{\nu_2} = B_i$, which is impossible, for the left hand side does not contain the element x of B_i.

If on the other hand $A_i \cup A_j = P$, we distinguish two possibilities:

1. The sets B_i, B_j are empty. If A_i and A_j have $p-1$ elements, the statement is true. If not, we may suppose A_i has at most $p-2$ elements. Let $P_{\nu_i} \supset A_{\nu_i}$ and $P_{\nu_j} \supset A_{\nu_j}$ be sets of $p-1$ elements. Then $P_{\nu_i} \cap P_{\nu_j}$ has $p-2$ elements. According to $(**)$ there is a set $A_k \subset P_{\nu_i} \cap P_{\nu_j}$ with the same number of elements as A_i, and applying $(**)$ for P_{ν_j} this implies that A_k and A_j, so A_i, A_k and A_j all have the same number of elements.

2. The set B_i is not empty. Let x, y, z be three elements of P satisfying the following requirements: $x \in A_i$, $x \notin A_j$, $y \in B_i$, $z \in A_j$ and $z \notin A_i \cup B_i$. (Such elements exist by condition 2 of the lemma). The set $P_\nu = P - y$ contains A_i, so there is an index $k \in I$ such that $A_k = A_i - x + z$, for this set is in P_ν and does not meet B_i. It is clear that $y \notin A_k \cup A_i$ and $x \notin A_k \cup A_j$, so neither $A_k \cup A_i$ nor $A_k \cup A_j$ is the set P. Consequently by the preceding argument, A_i, A_j and A_k all have the same number of elements and $B_i = B_j = B_k$.

This shows that if equality holds in $(*)$, then $B_i = B$ for all $i \in I$ and the sets A_i are certain q-element subsets of $P-B$. Since equality does hold in $(*)$ it must be that the sets A_i are all the q-element subsets of $P-B$.

This suffices to complete the proof of the lemma, by induction.

Proof of the theorem. Let G be a $(k+l)$-saturated generalized graph of n vertices. We denote by S_α $(\alpha \in A)$ the various k-tuples belonging to G and by N_β $(\beta \in B)$ the remaining k-tuples of vertices. The vertices not belonging to the k-tuple S_α will be denoted by $\bar{S}_\alpha$. Since G is saturated we can certainly choose, for each set N_β, at least one set K_β of $k+l$ vertices all k-tuples of which belong to the graph G except the k-tuple N_β. Suppose, therefore, that $N_\beta, S_{\alpha_1}, \dots, S_{\alpha_s}$ are the k-tuples of K_β. If S_{α_i} has m vertices in common with N_β, then we say that N_β assigns a weight $\dfrac{1}{\binom{n-l-m}{k-m}}$ to S_{α_i}, for $\alpha_i \in A$ and $\beta \in B$.

(a) Let us denote by W_0 the weight assigned by N_β to the set of k-tuples of K_β. It is clear that W_0 is independent of β and G, and that each N_β assignes a weight of at least W_0 to the set of k-tuples of G.

(b) We shall show that the k-tuples N_β altogether assign a weight of at most 1 to an arbitrary S_α. Suppose S_α has a weight assigned to it by $N_{\beta_1}, \dots, N_{\beta_r}$ and let $A_i = \bar{S}_\alpha \cap N_\beta$ for $i = 1, 2, \dots, r$. Then A_i is the set of vertices of N_{β_i} which are not in S_α.

(c) Let B_i denote the set of vertices of K_{β_i} which are neither in N_{β_i} nor in S_α. Then $A_i \cap B_i = \emptyset$, and there are l vertices in $A_i \cup B_i$. $A_j \not\subset A_i \cup B_i$ if $i \neq j$, for N_{β_i} is the only k-tuple of K_{β_i} not belonging to G, and $A_j \subset A_i \cup B_i$ would imply $N_{\beta_j} \subset K_{\beta_i}$.

If we let m_i denote the number of elements of A_i, then to prove statement (b) we need only show that

$$\sum_{i \in I} \frac{1}{\binom{(n-l)-(k-m_i)}{k-(k-m_i)}} \leqq 1, \quad \text{where} \quad I = (1, 2, \dots, r). \tag{3}$$

But statement (c) means that the sets A_i and B_i, for $i \in I$, satisfy the conditions of our lemma with $P=\bar{S}_\alpha, p = n-k$, $a_i=m_i$, and $b_i = l-m_i$. Hence,

$$\sum_{i\in I} \frac{1}{\binom{(n-k)-(l-m_i)}{m_i}} = \sum_{i\in I} \frac{1}{\binom{(n-k)-(k-m_i)}{k-(k-m_i)}} \leqq 1.$$

This proves statement (b).

Let there be t k-tuples S_α in G. Then there are $\binom{n}{k}-t$ k-tuples N_β. It follows from (a) that the k-tuples N_β altogether assign a weight of at least $\left(\binom{n}{k}-t\right)W_0$ to all the k-tuples S_α. From (b) it follows that the total weight assigned to the k-tuples S_α is at most t. Hence,

$$t \geqq \left(\binom{n}{k}-t\right)W_0,$$

or

$$t \geqq \frac{W_0\binom{n}{k}}{1+W_0}.$$

Equality holds here if and only if each N_β assignes a weight exactly W_0 and if each S_α is assigned the weight 1. It is easy to check that this is true for the graph $M(n,k,l)$. Therefore, t is at least as large as the number of k-tuples in $M(n,k,l)$, i. e. $t \geqq t(n,k,l)$.

If $t=t(n,k,l)$ for a graph G, then each N_β must assign a weight exactly W_0 and equality must hold in (3) for all $\alpha \in A$. Referring to the lemma this implies that the following two statements hold:

(4) For each k-tuple N_β not belonging to G there is exactly one set K_β of $k+l$ vertices such that N_β is the only k-tuple of K_β, since otherwise N_β would assign a weight more than W_0 to the set of S_α's.

(5) If $S_\alpha=(x_1, \dots, x_k)$ is an arbitrary k-tuple of G, there is a set of j vertices $(0 \leqq j \leqq l-1)$: $T = (x_{k+1}, \dots, x_{k+j})$ such that if $y_{k+j+1}, y_{k+j+2}, \dots, y_{k+l}$ are arbitrary vertices of G $(x_\mu \neq y_\nu)$, then the set $(x_1, \dots, x_{k+j}, y_{k+j+1}, \dots, y_{k+l})$ contains only one k-tuple N_β not belonging to G. Fort his k-tuple N_β: $N_\beta \cap T = \emptyset$ and $S_\alpha \cap N_\beta$ has $k+j-l$ elements.

Suppose $t=t(n,k,l)$ for a graph G, we shall prove $G=M(n,k,l)$. In the following S_α and N_β will mean k-tuples belonging and not belonging to G respectively, S_{α_i} and N_{β_i} will denote special k-tuples of the corresponding type.

(6) If $l=1$, (4) and (5) imply that for any set of $k+1$ vertices of G either all the k-tuples of this set are N_β's or only one of them; moreover for any k-tuple N_β there is only one set of $k+1$ vertices containing N_β, all the other k-tuples of which are S_α's; consequently for other sets of $k+1$ vertices containing N_β all the k-tuples of these sets are N_β's.

Let $K=(x_1, x_2, \dots, x_{k+1})$ be a set containing both S_α and N_β k-tuples: $(x_1, x_2, \dots, x_k)=S_{\alpha_1}$ and $(x_2, x_3, \dots, x_{k+1})=N_{\beta_1}$. Let y be an arbitrary vertex

$(y \neq x_i)$. By (6) all the k-tuples of the set $Y=(y, x_2, \ldots, x_{k+1})$ are N_β's. This implies that if S_{α_2} is an arbitrary k-tuple of K belonging to G, then all the k-tuples of $S_{\alpha_2}+y$ are S_α's, except the k-tuple in Y. From this it follows that if a k-tuple of $Y+x_1$ contains x_1, it is a k-tuple S_α, otherwise it is a k-tuple N_β. By using this method repeatedly we get that in G all the k-tuples containing x_1 are S_α's, and all the others N_β's, i. e. $G=M(n, k, 1)$.

If $l \geqq 2$, let $S_{\alpha_1}=(x_2, x_3, \ldots, x_{k+1})$ be an arbitrary k-tuple in $K_{\beta_1}= =(x_1, x_2, \ldots, x_{k+l})$, having $k-1$ common vertices with $N_{\beta_1}=(x_1, x_2, \ldots, x_k)$. To prove $G=M(n, k, l)$ we must show that no N_β meets $K_{\beta_1}-N_{\beta_1}=(x_{k+1}, \ldots, x_{k+l})$. In order to verify this it is sufficient to show, that for an arbitrary vertex x_0 $(x_0 \neq x_1, \ldots, x_{k+l})$ $(x_0, x_2, \ldots, x_k)$ is the k-tuple of $(x_0, x_2, x_3, \ldots, x_{k+l}) = K_{\beta_2}$ not belonging to G.

Suppose that for a vertex x_0 this is not true. Appealing to (5) K_{β_2} contains a k-tuple, not belonging to G, and this k-tuple does not meet the set $(x_{k+2}, \ldots, x_{k+l})$. By a simple change of notation we can obtain that $(x_0, x_3, x_4, \ldots, x_{k+1})=N_{\beta_2}$ is this k-tuple. Then $(x_3, x_4, \ldots, x_{k+2}) = S_{\alpha_2} \subset K_{\beta_1}$ is a k-tuple of G, $S_{\alpha_2} \subset K_{\beta_2}$, and so by (5) it has the same number of common vertices with N_{β_1} as with N_{β_2}. This is a contradiction since S_{α_2} has $k-2$ and $k-1$ common vertices with the sets N_{β_1} and N_{β_2} respectively. This suffices to complete the proof of theorem 1.

A generalized graph G is said to be *p-critical* if the smallest number of vertices that can represent all the k-tuples of G is p, but upon omitting any k-tuple the remaining k-tuples can be represented by $p-1$ vertices. The following is an immediate consequence of theorem 1.

THEOREM 2. *A p-critical generalized graph can have at most* $\binom{p+k-1}{k}$ *k-tuples and the only p-critical graphs with this many k-tuples consist of a complete* $(p+k-1)$*-graph and isolated vertices.*

PROOF. If G has n vertices and is p-critical, then it is easily seen that the complementary graph of G is $(n-p+1)$-saturated. The result now follows from theorem 1.

REMARK. In proving theorem 1 we actually proved the following result: If G has n vertices and the addition of any new k-tuple increases the number of complete $(k+l)$-graphs in G, then G has at least $t(n, k, l)$ k-tuples with equality holding only if $G=M(n, k, l)$.

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