7. Sunflowers

One of most beautiful results in extremal set theory is the so-called *Sunflower Lemma* discovered by Erdős and Rado (1960) asserting that in a sufficiently large uniform family, some highly regular configurations, called "sunflowers," must occur, regardless of the size of the universe. In this chapter we will consider this result as well as some of its modifications and applications.

7.1 The sunflower lemma

A *sunflower* (or Δ-*system*) with k *petals* and a *core* Y is a collection of sets $S_1, \ldots, S_k$ such that $S_i \cap S_j = Y$ for all $i \neq j$; the sets $S_i - Y$ are petals, and we require that none of them is empty. Note that a family of pairwise disjoint sets is a sunflower (with an empty core).

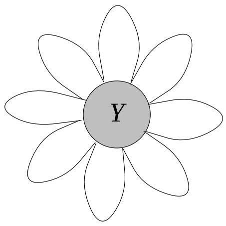

Fig. 7.1. A sunflower with 8 petals

Sunflower Lemma. *Let $\mathcal{F}$ be family of sets each of cardinality s. If $|\mathcal{F}| > s!(k-1)^s$ then $\mathcal{F}$ contains a sunflower with k petals.*

Proof. We proceed by induction on s. For $s = 1$, we have more than $k - 1$ points (disjoint 1-element sets), so any k of them form a sunflower with k petals (and an empty core). Now let $s \geqslant 2$, and take a maximal family $\mathcal{A} = \{A_1, \ldots, A_t\}$ of pairwise disjoint members of $\mathcal{F}$.

If $t \geqslant k$, these sets form a sunflower with $t \geqslant k$ petals (and empty core), and we are done.

Assume that $t \leqslant k-1$, and let $B = A_1 \cup \cdots \cup A_t$. Then $|B| \leqslant s(k-1)$. By the maximality of $\mathcal{A}$, the set B intersects every member of $\mathcal{F}$. By the pigeonhole principle, some point $x \in B$ must be contained in at least

$$\frac{|\mathcal{F}|}{|B|} > \frac{s!(k-1)^s}{s(k-1)} = (s-1)!(k-1)^{s-1}$$

members of $\mathcal{F}$. Let us delete x from these sets and consider the family

$$\mathcal{F}_x \rightleftharpoons \{S - \{x\} \ : \ S \in \mathcal{F}, \, x \in S\}.$$

By the induction hypothesis, this family contains a sunflower with k petals. Adding x to the members of this sunflower, we get the desired sunflower in the original family $\mathcal{F}$. $\square$

It is not known if the bound $s!(k-1)^s$ is the best possible. Let $f(s,k)$ denote the least integer so that any s-uniform family of $f(s,k)$ sets contains a sunflower with k petals. Then

$$(k-1)^s < f(s,k) \leqslant s!(k-1)^s + 1. \tag{7.1}$$

The upper bound is the sunflower lemma, the lower bound is Exercise 7.2. The gap between the upper and lower bound for $f(s,k)$ is still huge (by a factor of $s!$).

Conjecture 1 (Erdős and Rado)**.** *For every fixed k there is a constant $C = C(k)$ such that $f(s,k) < C^s$.*

The conjecture remains open even for $k = 3$ (note that in this case the sunflower lemma requires at least $s!2^s \approx s^s$ sets). Several authors have slightly improved the bounds in (7.1). In particular, J. Spencer has proved

$$f(s,3) \leqslant \mathrm{e}^{c\sqrt{s}} s!.$$

For s fixed and k sufficiently large, Kostochka et al. (1999) have proved

$$f(s,k) \leqslant k^s \left(1 + ck^{-2^{-s}}\right),$$

where c is a constant depending only on s.

But the proof or disproof of the conjecture is nowhere in sight.

A family $\mathcal{F} = \{S_1, \ldots, S_m\}$ is called a *weak Δ-system* if there is some λ such that $|S_i \cap S_j| = \lambda$ whenever $i \neq j$. Of course, not every such system is a sunflower: in a Δ-system it is enough that all the cardinalities of mutual intersections coincide whereas in a sunflower we require that these intersections all have the same elements. However, the following interesting result due to M. Deza states that if a weak Δ-system has many members then it is, in fact, "strong," i.e., forms a sunflower. We state this result without proof.

Theorem 7.1 (Deza 1973)**.** *Let $\mathcal{F}$ be an s-uniform weak Δ-system. If $|\mathcal{F}| \geqslant s^2 - s + 2$ then $\mathcal{F}$ is a sunflower.*