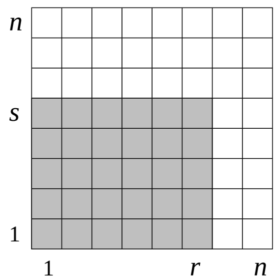
We will be interested in the length of the *longest* increasing and decreasing subsequences of A. It is intuitively plausible that there should be some kind of tradeoff between these lengths. If the longest increasing subsequence is short, say has length s, then *any* subsequence of A of length $s + 1$ must contain a pair of decreasing elements, so there are lots of pairs of decreasing elements. Hence, we would expect the longest decreasing sequence to be large. An extreme case occurs when $s = 1$. Then the whole sequence A is decreasing.

How can we quantify the feeling that the length of *both*, longest increasing and longest decreasing subsequences, cannot be small? A famous result of Erdős and Szekeres (1935) gives an answer to this question and was one of the first results in extremal combinatorics.

Theorem 4.4 (Erdős–Szekeres 1935). *Let $A = (a_1, \ldots, a_n)$ be a sequence of n different real numbers. If $n \geqslant sr + 1$ then either A has an increasing subsequence of $s + 1$ terms or a decreasing subsequence of $r + 1$ terms (or both).*

Proof (due to Seidenberg 1959). Associate to each term a_i of A a pair of "scores" (x_i, y_i) where x_i is the number of terms in the longest *increasing* subsequence *ending* at a_i, and y_i is the number of terms in the longest *decreasing* subsequence *starting* at a_i. Observe that no two terms have the same score, i.e., that $(x_i, y_i) \neq (x_j, y_j)$ whenever $i \neq j$. Indeed, if we have $\cdots a_i \cdots a_j \cdots$, then either $a_i < a_j$ and the longest increasing subsequence ending at a_i can be extended by adding on a_j (so that $x_i < x_j$), or $a_i > a_j$ and the longest decreasing subsequence starting at a_j can be preceded by a_i (so that $y_i > y_j$).

Now make a grid of n^2 pigeonholes:



Place each term a_i in the pigeonhole with coordinates (x_i, y_i). Each term of A can be placed in some pigeonhole, since $1 \leqslant x_i, y_i \leqslant n$ for all $i = 1, \ldots, n$. Moreover, no pigeonhole can have more than one term because $(x_i, y_i) \neq (x_j, y_j)$ whenever $i \neq j$. Since $|A| = n \geqslant sr + 1$, we have more items than the pigeonholes shaded in the above picture. So some term a_i will lie outside this shaded region. But this means that either $x_i \geqslant s + 1$ or $y_i \geqslant r + 1$ (or both), exactly what we need. □