29.3.2 Many players: the matrix product problem

One of the highest known lower bounds for the communication complexity of k-party games has the form $\Omega\left(4^{-k}\log_2 n\right)$. This bound was proved by Babai, Nisan, and Szegedy (1992) for so-called *generalized inner product* function GIP_n. In this case each X_i consists of all 0-1 (column) vectors of length $m = \lfloor \log_2 n \rfloor$; hence, the input space X consists of all binary $m \times k$ matrices. Given such a matrix $x \in X$, $GIP_n(x) = 1$ if and only if the number of all-1 rows in x is odd. Grolmusz (1994) has shown that this lower bound for GIP_n is almost optimal (see Exercise 29.6 and 29.7).

Similar lower bounds for other explicit functions were obtained by Chung (1990), Chung and Tetali (1993), Babai, Hayes and Kimmel (1998), Raz (2000), and other authors (see Babai 1997 for a survey). However, so far, no non-trivial lower bound is known for the case when the number of players k is larger than $\log_2 \log_2 n$.

As we already mentioned in Sect. 20.9, lower bounds on the multi-party communication complexity of explicit functions $F : X \to \{0,1\}$ can be obtained by bounding the discrepancy of the corresponding functions $f : X \to \{-1,1\}$ defined by $f(x) = 1 - 2 \cdot F(x)$. Now we explain this.

Recall that a subset $T \subseteq X$ is a *cylinder* in the ith dimension if membership in T_i does not depend on the ith coordinate. A subset $T \subseteq X$ is a *cylinder intersection* if it is an intersection $T = T_1 \cap T_2 \cap \cdots \cap T_k$, where T_i is a cylinder in the ith dimension. The *discrepancy* of a function $f : X \to \{-1,1\}$ on a set T is $\mathrm{disc}_T(f) = \left(\sum_{x\in T} f(x)\right) / |X|$. The *discrepancy* $\mathrm{disc}(f)$ of f is the maximum, over all cylinder intersections T, of the absolute value $|\mathrm{disc}_T(f)|$.

It can be shown (see Exercise 29.9) that a set T is a cylinder intersection if and only if it does not separate a sphere from its center, i.e., if for every sphere S around a vector x, $S \subseteq T$ implies $x \in T$. Thus, a coloring $c : X \to \{1, \ldots, r\}$ is legal for a given function $F : X \to \{0,1\}$ if and only if each color class $T = c^{-1}(i)$ is a cylinder intersection and the function F is constant on T. Since this last event is equivalent to $|\mathrm{disc}_T(f)| = |T|/|X|$, no color class can have more than $|X| \cdot \mathrm{disc}(f)$ vectors. This implies that we need at least $1/\mathrm{disc}(f)$ colors, and by Proposition 29.6,

$$C_k(F) \geqslant \log_2 \chi_k(F) \geqslant \log_2 \left(1/\mathrm{disc}(f)\right).$$

An explicit function $f : X \to \{-1,1\}$ (the matrix product function), for which

$$\mathrm{disc}(f) \leqslant \left(\frac{k-1}{\sqrt{\log_2 n}}\right)^{1/2^k},$$

was described in Sect. 20.9.1 (see Theorem 20.15). Hence, for the corresponding $F(x) = (1 - f(x))/2$, we have

$$C_k(F) = \Omega\left(2^{-k}\log_2 n\right).$$