Definition 27.2. Let $r, k, s_1, \ldots, s_r$ be given positive integers, $s_1, \ldots, s_r \geqslant k$. Then $R_r(k; s_1, \ldots, s_r)$ denotes the smallest number n with the property that, if the k-subsets of an n-set are colored with r colors $1, \ldots, r$, then for some $i \in \{1, \ldots, r\}$, there is an s_i-set, all of whose k-subsets have color i. If $s_1 = s_2 = \ldots = s_r = s$ then this number is denoted by $R_r(k; s)$.

Thus, the pigeonhole principle states that $R_r(1; 2) = 1 + r$ and Proposition 27.1 that $R_r(1; s) = 1 + r(s - 1)$. These are 1-dimensional results since $k = 1$.

27.2 Ramsey's theorem for graphs

The 2-dimensional case ($k = 2$) corresponds to coloring the edges of a graph. Moreover, if we consider 2-colorings ($r = 2$) then the corresponding Ramsey number is denoted by $R(s, t)$, i.e., $R(s, t) \rightleftharpoons R_2(2; s, t)$.

To warm-up let us consider the following simple game. Mark six points on the paper, no three in line. There are two players; one has a Red pencil the other Blue. Each player's turn consists in drawing a line with his/her pencil between two of the points which haven't already been joined. (The crossing of lines is allowed). The player's goal is to create a triangle in his/her color. If you try to play it with a friend, you will notice that it always end in a win for one player: a draw is not possible. Is this really so? In terms of Ramsey numbers, we ask if $R(3, 3) \leqslant 6$: we have $r = 2$ colors, edges are k-sets with $k = 2$ and we are looking for a monochromatic 3-set. Prove that indeed, $R(3, 3) = 6$. (Hint: see Fig. 27.1.)

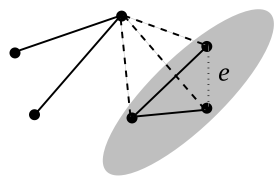

Fig. 27.1. What is the color of e?

You have just shown that the number $R(s, t)$ exists if $s = t = 3$. This is a very special case of the well-known version of Ramsey's theorem for graphs, which says that $R(s, t)$ exists for any natural numbers s and t.

Let $G = (V, E)$ be an undirected graph. A subset $S \subseteq V$ is a *clique* of G if any two vertices of S are adjacent. Similarly, a subset $T \subseteq V$ is an *independent set* of G if no two vertices of T are adjacent in G.

Theorem 27.3. *For any natural numbers s and t there exists a natural number $n = R(s, t)$ such that in any graph on n or more vertices, there exists either a clique of s vertices or an independent set of t vertices.*

Proof. To prove the existence of the desired number $n = R(s,t)$, it is sufficient to show, by induction on $s+t$, that $R(s,t)$ is bounded. For the base case, it is easy to verify that $R(1,t) = R(s,1) = 1$. For $s > 1$ and $t > 1$, let us prove that

$$R(s,t) \leqslant R(s,t-1) + R(s-1,t). \tag{27.1}$$

Let $G = (V,E)$ be a graph on $n = R(s,t-1) + R(s-1,t)$ vertices. Take an arbitrary vertex $x \in V$, and split $V - \{x\}$ into two subsets S and T, where each vertex of S is nonadjacent to x and each vertex of T is adjacent to x (see Fig. 27.2). Since

$$R(s,t-1) + R(s-1,t) = |S| + |T| + 1,$$

we have either $|S| \geqslant R(s,t-1)$ or $|T| \geqslant R(s-1,t)$.

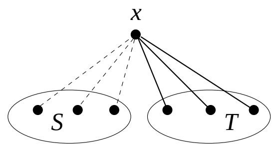

Fig. 27.2. Splitting the graph into neighbors and non-neighbors of x

Let $|S| \geqslant R(s,t-1)$, and consider the induced subgraph $G[S]$ of G: this is a graph on vertices S, in which two vertices are adjacent if and only if they are such in G. Since the graph $G[S]$ has at least $R(s,t-1)$ vertices, by the induction hypothesis, it contains either a clique on s vertices or an independent set of $t-1$ vertices. Moreover, we know that x is not adjacent to any vertex of S in G. By adding this vertex to S, we conclude that the subgraph $G[S \cup \{x\}]$ (and hence, the graph G itself) contains either a clique of s vertices or an independent set of t vertices.

The case when $|T| \geqslant R(s-1,t)$ is analogous. □

The recurrence (27.1) implies (see Exercise 27.6)

$$R(s,t) \leqslant \binom{s+t-2}{s-1}. \tag{27.2}$$

The lower bound on $R(t,t)$ of order $t2^{t/2}$ was proved in Chap. 18 (Theorem 18.1) using the probabilistic method. Thus,

$$c_1 t 2^{t/2} \leqslant R(t,t) \leqslant \binom{2t-2}{t-1} \sim c_2 4^t/\sqrt{t}. \tag{27.3}$$

The gap is still large, and in recent years, relatively little progress has been made. Tight bounds are known only for $s = 3$:

$$c_1 \frac{t^2}{\log t} \leqslant R(3,t) \leqslant c_2 \frac{t^2}{\log t}.$$