Our goal is to show that $|X| \leqslant k$.

Note that, for each $x \in X$ we can choose $Y_x \subseteq V_2$ so that $1 \leqslant |Y_x| \leqslant r$, $x \in \Gamma(Y_x)$ and

$$|\Gamma(Y_x) - X| = r - |Y_x|;$$

otherwise X could be replaced by $X - \{x\}$, contradicting the minimality of X. We will apply Lemma 2.6 to the bipartite graph $G' = (X, V_2, F)$, where

$$F = \{(x, y) : y \in Y_x\}.$$

All we have to do is to show that the hypothesis of the lemma is satisfied by the function (here $N(y)$ is the set of neighbors of y in the original graph G):

$$f(y) \rightleftharpoons \frac{|N(y)|}{r},$$

because then

$$|X| \leqslant \sum_{y \in V_2} f(y) = \frac{1}{r} \sum_{y \in V_2} |N(y)| = \frac{|E|}{r} < k + 1.$$

Consider an edge $(x, y) \in F$; we have to show that $d(y) \leqslant d(x) \cdot f(y)$, where

$$d(x) = |Y_x| \quad \text{and} \quad d(y) = |\{x \in X : y \in Y_x\}|$$

are the degrees of x and y in the graph $G' = (X, V_2, F)$. Now, $y \in Y_x$ implies $\Gamma(Y_x) \subseteq N(y)$, which in its turn implies

$$|N(y) - X| \geqslant |\Gamma(Y_x) - X| = r - |Y_x|;$$

hence

$$\begin{aligned} d(y) &\leqslant |N(y) \cap X| = |N(y)| - |N(y) - X| \\ &\leqslant |N(y)| - r + |Y_x| = r \cdot f(y) - r + d(x), \end{aligned}$$

and so

$$\begin{aligned} d(x) \cdot f(y) - d(y) &\geqslant d(x) \cdot f(y) - r \cdot f(y) + r - d(x) \\ &= (r - d(x)) \cdot (1 - f(y)) \geqslant 0. \end{aligned}$$

□

2.3 Density of 0-1 matrices

Let H be an $m \times n$ 0-1 matrix. We say that H is *α-dense* if at least an α-fraction of all its mn entries are 1's. Similarly, a row (or column) is *α-dense* if at least an α-fraction of all its entries are 1's.

The next result says that any dense 0-1 matrix must either have one "very dense" row or there must be many rows which are still "dense enough."

Lemma 2.7 (Grigni–Sipser 1995)**.** *If H is 2α-dense then either*
(a) *there exists a row which is $\sqrt{\alpha}$-dense, or*
(b) *at least $\sqrt{\alpha} \cdot m$ of the rows are α-dense.*

Note that $\sqrt{\alpha}$ is larger than α when $\alpha < 1$.

Proof. Suppose that the two cases do not hold. We calculate the density of the entire matrix. Since (b) does not hold, less than $\sqrt{\alpha} \cdot m$ of the rows are α-dense. Since (a) does not hold, each of these rows has less than $\sqrt{\alpha} \cdot n$ 1's; hence, the fraction of 1's in α-dense rows is strictly less than $(\sqrt{\alpha})(\sqrt{\alpha}) = \alpha$. We have at most m rows which are not α-dense, and each of them has less than αn 1's. Hence, the fraction of 1's in these rows is also less than α. Thus, the total fraction of 1's in the matrix is less than 2α, a contradiction with the 2α-density of H. □

Now consider a slightly different question: if H is α-dense, how many of its rows *or* columns are "dense enough"? The answer is given by the following general estimate due to Johan Håstad. This result appeared in the paper of Karchmer and Wigderson (1990) and was used to prove that the graph connectivity problem cannot be solved by monotone circuits of logarithmic depth.

For a subset H of a universe Ω, its *density* is the fraction $\mu(H) \rightleftharpoons \frac{|H|}{|\Omega|}$.

Suppose that our universe is a Cartesian product $\Omega = A_1 \times \cdots \times A_k$ of some finite sets $A_1, \ldots, A_k$. Hence, elements of Ω are strings $a = (a_1, \ldots, a_k)$ with $a_i \in A_i$. Given a set $H \subseteq \Omega$ of such strings and a point $b \in A_i$, we denote by $H_{i \to b}$ the set of all strings $(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k)$ for which the extended string $(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_k)$ belongs to H.

We stress that the density of a set depends on the size of the underlying universe Ω. Since the universe for $H_{i \to b}$ is $|A_i|$ times smaller than that for H, we have $\mu(H_{i \to b}) = |A_i| \cdot \mu\{a \in H : a_i = b\}$.

Lemma 2.8 (J. Håstad)**.** *Let $B = B_1 \times \cdots \times B_k$, where*

$$B_i \rightleftharpoons \left\{ b \in A_i \, : \, \mu\left(H_{i \to b}\right) \geqslant \frac{\mu(H)}{2k} \right\}.$$

Then $\mu(B) \geqslant \frac{1}{2}\mu(H)$.

Proof. We have $\mu(B) \geqslant \mu(H \cap B) = \mu(H) - \mu(H - B)$, where

$$\begin{aligned} \mu(H - B) &\leqslant \sum_{i=1}^{k} \sum_{b \notin B_i} \mu\{a \in H \, : \, a_i = b\} = \sum_{i=1}^{k} \sum_{b \notin B_i} \frac{\mu\left(H_{i \to b}\right)}{|A_i|} \\ &< \sum_{i=1}^{k} \sum_{b \notin B_i} \frac{\mu(H)}{2k \cdot |A_i|} \leqslant \sum_{i=1}^{k} \frac{\mu(H)}{2k} = \frac{\mu(H)}{2}. \end{aligned}$$

□

Corollary 2.9. *In any 2α-dense matrix either a $\sqrt{\alpha}$-fraction of its rows or a $\sqrt{\alpha}$-fraction of its columns (or both) are $(\alpha/2)$-dense.*

Proof. The case $k = 2$ of Lemma 2.8 corresponds to 0-1 matrices, and in this case the lemma says that $\mu(B_i) \geqslant \left(\frac{\mu(H)}{2}\right)^{1/2}$ for some $i \in \{1, 2\}$. □

Exercises

2.1. Let $A_1, \ldots, A_m$ be subsets of an n-element set such that $|A_i \cap A_j| \leqslant t$ for all $i \neq j$. Prove that $\sum_{i=1}^{m} |A_i| \leqslant n + t \cdot \binom{m}{2}$.

2.2.$^{(!)}$ Let $A = (a_{ij})$ be an $n \times n$ matrix ($n \geqslant 4$). The matrix is filled with integers and each integer appears exactly twice. Show that there exists a permutation π of $\{1, \ldots, n\}$ such that all the numbers $a_{i,\pi(i)}$, $i = 1, \ldots, n$ are distinct. (Such a permutation π is also called a *Latin transversal* of A.)

> *Hint*: Look at how many pairs of entries are "bad," i.e., contain the same number, and show that strictly less than $n!$ of all permutations can go through such pairs.

2.3.$^{-}$ Let $\mathcal{F}$ be a family of m subsets of a finite set X. For $x \in X$, let $p(x)$ be the number of pairs (A, B) of sets $A \neq B \in \mathcal{F}$ such that either $x \in A \cap B$ or $x \notin A \cup B$. Prove that $p(x) \geqslant m^2/2$ for every $x \in X$.

> *Hint*: Let $d(x)$ be the degree of x in $\mathcal{F}$, and observe that $p(x) = d(x)^2 + (m - d(x))^2$.

2.4.$^{+}$ Let $\mathcal{F}$ be a family of nonempty subsets of a finite set X that is closed under union (i.e., $A, B \in \mathcal{F}$ implies $A \cup B \in \mathcal{F}$). Prove or give a counterexample: there exists $x \in X$ such that $d(x) \geqslant |\mathcal{F}|/2$. (Open conjecture, due to Peter Frankl.)

2.5.$^{-}$ A *projective plane* of order $r - 1$ is a family of $n = r^2 - r + 1$ r-element subsets (called *lines*) of an n-element set of points such that each two lines intersect in precisely one point and each point belongs to precisely r lines (cf. Sect. 13.4). Use this family to show that the bound given by Corrádi's lemma (Lemma 2.1) is optimal.

2.6. Theorem 2.4 gives a sufficient condition for a bipartite graph with parts of the same size n to contain an $a \times a$ clique. Extend this result to not necessarily balanced graphs. Let $k_{a,b}(m, n)$ be the minimal integer k such that any bipartite graph with parts of size m and n and more than k edges contains at least one $a \times b$ clique. Prove that for any $0 \leqslant a \leqslant m$ and $0 \leqslant b \leqslant n$,

$$k_{a,b}(m, n) \leqslant (a - 1)^{1/b}(n - b + 1)m^{1-1/b} + (b - 1)m.$$