Lemma 15.2 (Alon 1990a)**.** *Let* $v_i = (v_{i1}, v_{i2}, \ldots, v_{in})$*,* $1 \leqslant i \leqslant k$*, be* k *mutually orthogonal vectors, where* $v_{ij} \in \{-1, +1\}$*, and let* $c_1, c_2, \ldots, c_k$ *be* k *reals, not all zero. Then the vector* $y = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$ *has at least* n/k *nonzero entries.*

Proof. Assume, without loss of generality, that $|c_1| = max_{1 \leqslant i \leqslant k} |c_i|$. Put $y = (y_1, y_2, \ldots, y_n)$ and let s be the number of nonzero entries in y, i.e., $s = |S|$ where $S \rightleftharpoons \{j : y_j \neq 0\}$. Also let $|y|$ stand for the vector $(|y_1|, \ldots, |y_n|)$ of the absolute values of entries of y. Since vectors $v_1, \ldots, v_k$ are mutually orthogonal, we have

$$\begin{aligned} kc_1^2 n &\geqslant \sum_{i=1}^{k} c_i^2 n = \sum_{i=1}^{k} \langle c_i v_i, c_i v_i \rangle = \left\langle \sum_{i=1}^{k} c_i v_i, \sum_{i=1}^{k} c_i v_i \right\rangle = \langle y, y \rangle \\ &= \sum_{j=1}^{n} y_j^2 = \sum_{j \in S} |y_j|^2 = \frac{1}{s} \Big(\sum_{j \in S} 1 \Big) \Big(\sum_{j \in S} |y_j|^2 \Big) \geqslant \frac{1}{s} \Big(\sum_{j \in S} |y_j| \Big)^2, \end{aligned}$$

where the last inequality follows from Cauchy–Schwarz inequality (14.3). On the other hand, since v_1 is orthogonal to all the vectors $v_2, \ldots, v_k$,

$$\begin{aligned} \sum_{j=1}^{n} |y_j| &\geqslant \sum_{j=1}^{n} y_j v_{1j} = \sum_{j=1}^{n} \sum_{i=1}^{k} c_i v_{ij} v_{1j} \\ &= \sum_{i=1}^{k} c_i \sum_{j=1}^{n} v_{ij} v_{1j} = \sum_{i=1}^{k} c_i \langle v_i, v_1 \rangle = c_1 \langle v_1, v_1 \rangle = c_1 \cdot n. \end{aligned}$$

Substituting this estimate into the previous one we obtain $s \geqslant n/k$, as desired. □

This result gives us some new information about Hadamard matrices. A Hadamard matrix is a square $n \times n$ matrix H with entries in $\{-1, +1\}$ and with row vectors mutually orthogonal (and hence with column vectors mutually orthogonal). Thus, H has a maximal rank n. (Recall that the rank of a matrix is the minimal number of linearly independent rows.)

The following corollary from Alon's lemma says that not only the Hadamard matrix itself but also each of its large enough submatrices has maximal rank.

Corollary 15.3. *If* $t > (1 - 1/r)n$*, then every* $r \times t$ *sub-matrix* H' *of an* $n \times n$ *Hadamard matrix* H *has rank* r *(over the reals).*

Proof. Suppose this is false. Then there is a real nontrivial linear combination of the rows of H' that vanishes. But by Lemma 15.2 this combination, taken with the corresponding rows of H, has at least n/r nonzero entries, and at least one of these must appear in a column of H', a contradiction. □

Changing some entries of a real matrix by appropriate reals we can always reduce its rank. The *rigidity* of a matrix M is the function $R_M(r)$, which for a given r, gives the minimum number of entries of M which one has to change in order to reduce its rank to r or less. Due to its importance, the rigidity of Hadamard matrices deserves continuous attention. For an $n \times n$ Hadamard matrix H, Pudlák, Razborov, and Savický (1988) proved that $R_H(r) \geqslant \frac{n^2}{r^3 \log r}$. Alon's lemma implies that $R_H(r) \geqslant \frac{n^2}{r^2}$.

Corollary 15.4. *If less than $(n/r)^2$ entries of an $n \times n$ Hadamard matrix H are changed (over the reals) then the rank of the resulting matrix remains at least r.*

Proof. Split H into n/r submatrices with r rows in each. Since less than $(n/r)(n/r)$ of the entries of H are changed, in at least one of these $r \times n$ submatrices strictly less than n/r changes are made. Thus, there is an $r \times t$ submatrix, with $t > n - n/r$, in which no change has taken place. The result now follows from Corollary 15.3. □

15.1.3 Hadamard matrices

In the previous section we demonstrated how, using an elementary linear algebra, one can prove some non-trivial facts about the rank and rigidity of Hadamard matrices. In this section we will establish several other important properties of these matrices. Recall that a Hadamard matrix of order n is an $n \times n$ matrix with entries in $\{-1, +1\}$ whose rows (columns) are mutually orthogonal.

The first property is the *Lindsey Lemma*. Its proof can be found in Erdős and Spencer (1974). We present the neat proof given in Babai, Frankl and Simon (1986).

Lemma 15.5 (J. H. Lindsey). *Let H be an $n \times n$ Hadamard matrix and T be an arbitrary $a \times b$ sub-matrix of H. Then the difference between the number of $+1'$s and $-1'$s in T is at most $\sqrt{abn}$.*

Proof. Let $v_i = (v_{i1}, \ldots, v_{in})$ denote the ith row of H. Assume that T consists of its first a rows and b columns, and let

$$\alpha \rightleftharpoons \sum_{i=1}^{a} \sum_{j=1}^{b} v_{ij}.$$

We want to prove that $\alpha \leqslant \sqrt{abn}$.

Set $x = (1^b 0^{n-b})$ and consider the vector $y = (y_1, \ldots, y_n) \rightleftharpoons \sum_{i=1}^{a} v_i$. By the Cauchy–Schwarz inequality (14.3),

$$\alpha^2 = \langle x, y \rangle^2 \leqslant \|x\|^2 \|y\|^2 = b \cdot \|y\|^2.$$

Since H is Hadamard, the vectors v_i are orthogonal; hence

$$\|y\|^2 = \langle y, y \rangle = \left\langle \sum_{i=1}^{a} v_i, \sum_{i=1}^{a} v_i \right\rangle = \sum_{i=1}^{a} \sum_{j=1}^{a} \langle v_i, v_j \rangle = \sum_{i=1}^{a} \langle v_i, v_i \rangle = an.$$