**14.22**$^{(!)}$ (Ray-Chaudhuri–Wilson 1975). Prove the following uniform version of Theorem 14.13: if $A_1, \ldots, A_m$ is a $k$-uniform $L$-intersecting family of subsets of an $n$-element set, then $m \leqslant \binom{n}{s}$, where $s = |L|$.

*Sketch* (Alon–Babai–Suzuki 1991): Start as in the proof of Theorem 14.13 and define the same polynomials $f_1, \ldots, f_m$ of degree at most $s$. Associate with each subset $I$ of $\{1, \ldots, n\}$ of cardinality $|I| \leqslant s-1$ the following polynomial of degree at most $s$:

$$g_I(x) = \Big(\Big(\sum_{j=1}^{n} x_j\Big) - k\Big) \prod_{i \in I} x_i,$$

and observe that for any subset $S \subseteq \{1, \ldots, n\}$, $g_I(S) \neq 0$ if and only if $|S| \neq k$ and $S \supseteq I$. Use this property to show that the polynomials $g_I$ *together* with the polynomials $f_i$ are linearly independent. For this, assume

$$\sum_{i=1}^{m} \lambda_i f_i + \sum_{|I| \leqslant s-1} \mu_I g_I = 0$$

for some $\lambda_i, \mu_I \in \mathbb{R}$. Substitute $A_j$'s for the variables in this equation to show that $\lambda_j = 0$ for every $j = 1, \ldots, m$. What remains is a relation among the $g_I$. To show that this relation must be also trivial, assume the opposite and re-write this relation as $\mu_1 g_{I_1} + \cdots + \mu_t g_{I_t} = 0$ with all $\mu_i \neq 0$ and $|I_1| \geqslant |I_j|$ for all $j > 1$. Show that then $g_{I_1}(I_1) \neq 0$ and $g_{I_1}(I_j) = 0$ for all $j > 1$.

**14.23.** Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be subsets of an $n$-element set such that $|A_i \cap B_i|$ is odd for all $1 \leqslant i \leqslant m$, and $|A_i \cap B_j|$ is even for all $1 \leqslant i < j \leqslant m$. Show that then $m \leqslant n$.

**14.24**$^{(!)}$ (Frankl–Wilson 1981). Let $p$ be a prime, and $n = 4p - 1$. Consider the graph $G = (V, E)$ whose vertex set $V$ consists of all 0-1 vectors of length $n$ with precisely $2p - 1$ 1's each; two vectors are adjacent if and only if the Euclidean distance between them is $\sqrt{2p}$. Show that $G$ has no independent set of size larger than $\sum_{i=0}^{p-1} \binom{n}{i}$.

*Hint*: Use Exercise 14.8 to show that two vectors from $V$ are adjacent in $G$ precisely when they share $p - 1$ 1's in common, and apply Theorem 14.14.

*Comment*: This construction was used by Frankl and Wilson (1981) to resolve an old problem proposed by H. Hadwiger in 1944: how many colors do we need in order to color the points of the $n$-dimensional Euclidean space $\mathbb{R}^n$ so that each monochromatic set of points misses some distance? A set is said to *miss distance* $d$ if no two of its points are at distance $d$ apart from each other. Larman and Rogers (1972) proved that Hadwiger's problem reduces to the estimating the minimum number of colors $\chi(n)$ necessary to color the points of $\mathbb{R}^n$ such that pairs of points of unit distance are colored differently. The graph $G$ we just constructed shows that $\chi(n) \geqslant 2^{\Omega(n)}$ (see the next exercise). Kahn and Kalai (1993) used a similar construction to disprove another 60 years old and widely believed conjecture of K. Borsuk (1933) that every set of diameter one in $n$-dimensional real space $R^n$ can be partitioned in at most $n + 1$ disjoint pieces of smaller diameter. Kahn and Kalai presented an infinite sequence of examples where the minimum number of pieces grew as an exponential function of $\sqrt{n}$, rather than just as a linear function $n + 1$, as conjectured. Interested reader can find these surprising solutions in the book of Babai and Frankl (1992).