In cases when $V = \mathbb{F}^n$ for some field $\mathbb{F}$, there is a particularly elegant and explicit choice of $|\mathbb{F}|$ points in general position, using the so-called *moment curve*. This curve is defined as the range of the function

$$\mathbb{F} \ni a \mapsto m(a) \rightleftharpoons \left(1, a, a^2, \ldots, a^{n-1}\right) \in \mathbb{F}^n.$$

Lemma 14.5. *Let $V = \mathbb{F}^n$ and $|\mathbb{F}| \geqslant n$. Then the set of $|\mathbb{F}|$ vectors $m(a)$, $a \in \mathbb{F}$, is in general position.*

Proof. For n distinct elements $a_1, \ldots, a_n \in \mathbb{F}$, consider the determinant of the corresponding $n \times n$ matrix with rows $m(a_i)$. This determinant is known as a Vandermonde determinant (cf. Exercise 14.11); its value is

$$\prod_{1 \leqslant i < j \leq n} (a_j - a_i) \neq 0.$$

Therefore the rows $m(a_i)$ are linearly independent (consult Exercise 14.12 for this last conclusion). $\square$

Let us now look how the linear algebra argument works in concrete situations.

14.2 Spaces of incidence vectors

Suppose we are given a family $\mathcal{F}$ of sets satisfying some conditions. We want to know how many sets such a family can have. In some situations it is sufficient to associate sets to their incidence vectors and show that these vectors are linearly independent.

14.2.1 Fisher's inequality

Suppose that each two sets of our family share the same number of elements. How large can such a family be? The answer is given by a fundamental result of design theory – known as *Fisher's inequality.*

Theorem 14.6 (Fisher's inequality). *Let $A_1, \ldots, A_m$ be distinct subsets of $\{1, \ldots, n\}$ such that $|A_i \cap A_j| = k$ for some fixed $1 \leqslant k \leqslant n$ and every $i \neq j$. Then $m \leqslant n$.*

Proof. Let $v_1, \ldots, v_m \in \{0,1\}^n$ be incidence vectors of $A_1, \ldots, A_m$. By the linear algebra bound (Proposition 14.1) it is enough to show that these vectors are linearly independent over the reals. Assume the contrary, i.e., that the linear relation $\sum_{i=1}^m \lambda_i v_i = 0$ exists, with not all coefficients being zero. Obviously, $\langle v_i, v_j \rangle = |A_i|$ if $j = i$, and $\langle v_i, v_j \rangle = k$ if $j \neq i$. Consequently,

$$0 = \left(\sum_{i=1}^{m} \lambda_i v_i\right)\left(\sum_{j=1}^{m} \lambda_j v_j\right) = \sum_{i=1}^{m} \lambda_i^2 \langle v_i, v_i\rangle + \sum_{1\leqslant i\neq j\leqslant m} \lambda_i\lambda_j \langle v_i, v_j\rangle$$
$$= \sum_{i=1}^{m} \lambda_i^2 |A_i| + \sum_{1\leqslant i\neq j\leqslant m} \lambda_i\lambda_j k = \sum_{i=1}^{m} \lambda_i^2(|A_i| - k) + k\cdot\left(\sum_{i=1}^{m}\lambda_i\right)^2.$$

Clearly, $|A_i| \geqslant k$ for all i and $|A_i| = k$ for at most one i, since otherwise the intersection condition would not be satisfied. But then the right-hand is greater than 0 (because the last sum can vanish only if at least two of the coefficients λ_i are nonzero), a contradiction. □

This theorem was first proved by the statistician R. A. Fisher in 1940 for the case when $k = 1$ and all sets A_i have the same size (such configurations are known as balanced incomplete block designs). In 1948, de Bruijn and Erdős relaxed the uniformity condition for the sets A_i (see Theorem 13.4). This was generalized by R. C. Bose in 1949, and later by several other authors. But it was the two-page paper of Bose where the linear argument was first applied to solve a combinatorial problem. The general version, stated above, was first proved by Majumdar (1953); the proof we presented is a variation of a simplified argument found in Babai and Frankl (1992).

14.2.2 Inclusion matrices

In Chap. 11 we have proved a fundamental result concerning the Vapnik–Chervonenkis dimension (see Theorem 11.1). Let us restate this result in terms of set systems.

Let $\mathcal{F}$ be a family of subsets of an n-element set X. Such a family is (n,k)-*dense* if there is a subset $Y \subseteq X$ of cardinality $|Y| = k$ such that every subset Z of Y occurs as $Z = E \cap Y$ for some $E \in \mathcal{F}$.

Theorem 14.7. *Let $\mathcal{F}$ be a family of subsets of an n-element set X. If $\mathcal{F}$ has more than $\sum_{i=0}^{k-1}\binom{n}{i}$ members, then $\mathcal{F}$ is (n,k)-dense.*

We presented two proofs of this result. One was by induction on n and k, whereas the other was based on a striking observation made by Alon (1983) and Frankl (1983) that for results like this, it is enough to consider only those families which are downwards closed. Here we will give one more proof, based on linear independence. The advantage of this argument is that it can be easily modified to yield a similar result for uniform families (see Exercise 14.14).

Proof (due to Frankl and Pach 1983). Let $Y_1, Y_2, \ldots, Y_r$, $r = \sum_{i=0}^{k-1}\binom{n}{i}$, be an enumeration of all subsets of X of size at most $k-1$, and let $E_1, E_2, \ldots, E_s$ denote the members of $\mathcal{F}$. Define an $s \times r$ 0-1 matrix $M = (m_{ij})$ by

$m_{ij} = 1$ if and only if $E_i \supseteq Y_j$.