9.2.4 Union-free families

A family of sets $\mathcal{F}$ is called *r-union-free* if $A_0 \not\subseteq A_1 \cup A_2 \cup \cdots \cup A_r$ holds for all distinct $A_0, A_1, \ldots, A_r \in \mathcal{F}$. Thus, antichains are r-union-free for $r = 1$.

Let $T(n,r)$ denote the maximum cardinality of an r-union-free family $\mathcal{F}$ over an n-element underlying set. This notion was introduced by Kautz and Singleton (1964). They proved that

$$\Omega(1/r^2) \leqslant \frac{\log_2 T(n,r)}{n} \leqslant O(1/r).$$

This result was rediscovered several times in information theory, in combinatorics by Erdős, Frankl, and Füredi (1985), and in group testing by Hwang and Sós (1987). Dyachkov and Rykov (1982) obtained, with a rather involved proof, that

$$\frac{\log_2 T(n,r)}{n} \leqslant O(\log_2 r/r^2).$$

Recently, Ruszinkó (1994) gave a purely combinatorial proof of this upper bound. Shortly after, Füredi (1996) found a very elegant argument, and we present it below.

Theorem 9.14 (Füredi 1996)**.** *Let $\mathcal{F}$ be a family of subsets of an n-element underlying set X, and $r \geqslant 2$. If $\mathcal{F}$ is r-union-free then $|\mathcal{F}| \leqslant r + \binom{n}{t}$ where*

$$t \rightleftharpoons \left\lceil (n-r)/\binom{r+1}{2} \right\rceil.$$

That is,

$$\log_2 |\mathcal{F}|/n \leqslant O\left(\log_2 r/r^2\right).$$

Proof. Let $\mathcal{F}_t$ be the family of all members of $\mathcal{F}$ having their *own* t-subset. That is, $\mathcal{F}_t$ contains all those members $A \in \mathcal{F}$ for which there exists a t-element subset $T \subseteq A$ such that $T \not\subseteq A'$ for every other $A' \in \mathcal{F}$. Let $\mathcal{T}_t$ be the family of these t-subsets; hence $|\mathcal{T}_t| = |\mathcal{F}_t|$. Let $\mathcal{F}_0 \rightleftharpoons \{A \in \mathcal{F} : |A| < t\}$, and let $\mathcal{T}_0$ be the family of *all* t-subsets of X containing a member of $\mathcal{F}_0$, i.e.,

$$\mathcal{T}_0 \rightleftharpoons \{T : T \subseteq X, |T| = t \text{ and } T \supset A \text{ for some } A \in \mathcal{F}_0\}.$$

The family $\mathcal{F}$ is an antichain. This implies that $\mathcal{T}_t$ and $\mathcal{T}_0$ are disjoint. The family $\mathcal{F}_0$ is also an antichain, and since $t < n/2$, we know from Exercise 5.9 that $|\mathcal{F}_0| \leqslant |\mathcal{T}_0|$. Therefore,

$$|\mathcal{F}_0 \cup \mathcal{F}_t| \leqslant |\mathcal{T}_t| + |\mathcal{T}_0| \leqslant \binom{n}{t}. \tag{9.3}$$

It remains to show that the family

$$\mathcal{F}' \rightleftharpoons \mathcal{F} - (\mathcal{F}_0 \cup \mathcal{F}_t)$$

has at most r members. Note that $A \in \mathcal{F}'$ if and only if $A \in \mathcal{F}$, $|A| \geqslant t$ and for every t-subset $T \subseteq A$ there is an $A' \in \mathcal{F}$ such that $A' \neq A$ and $A' \supseteq T$.

We will use this property to prove that $A \in \mathcal{F}'$, $A_1, A_2, \ldots, A_i \in \mathcal{F}$ $(i \leqslant r)$ imply

$$|A - (A_1 \cup \cdots \cup A_i)| \geqslant t(r-i) + 1. \tag{9.4}$$

To show this, assume the opposite. Then the set $A - (A_1 \cup \cdots \cup A_i)$ can be written as the union of some $(r-i)$ t-element sets $T_{i+1}, \ldots T_r$. Therefore, A lies entirely in the union of $A_1, \ldots, A_i$ and these sets $T_{i+1}, \ldots, T_r$. But, by the choice of A, each of the sets T_j lies in some other set $A_j \in \mathcal{F}$ different from A. Therefore, $A \subseteq A_1 \cup \cdots \cup A_r$, a contradiction.

Now suppose that $\mathcal{F}'$ has more than r members, and take any $r+1$ of them $A_0, A_1, \ldots, A_r \in \mathcal{F}'$. Applying (9.4) we obtain

$$\begin{aligned}
\left|\bigcup_{i=0}^{r} A_i\right| &= |A_0| + |A_1 - A_0| + |A_2 - (A_0 \cup A_1)| + \cdots \\
&\quad + |A_r - (A_0 \cup A_1 \cup \cdots \cup A_{r-1})| \\
&\geqslant \big(tr+1\big) + \big(t(r-1)+1\big) + \big(t(r-2)+1\big) + \cdots + \big(t \cdot 0 + 1\big) \\
&= t \cdot \frac{r(r+1)}{2} + r + 1 = t\binom{r+1}{2} + r + 1.
\end{aligned}$$

By the choice of t, the right-hand side exceeds the total number of points n, which is impossible. Therefore, $\mathcal{F}'$ cannot have more than r distinct members. Together with (9.3), this yields the desired upper bound on $|\mathcal{F}|$. □

Exercises

9.1.$^{-}$ Let $\mathcal{F}$ be an antichain consisting of sets of size at most $k \leqslant n/2$. Show that $|\mathcal{F}| \leqslant \binom{n}{k}$.

9.2. Derive from Bollobás's theorem the following weaker version of Theorem 9.11. Let $A_1, \ldots, A_m$ be a collection of a-element sets and $B_1, \ldots, B_m$ be a collection of b-element sets such that $|A_i \cap B_i| = t$ for all i, and $|A_i \cap B_j| > t$ for $i \neq j$. Then $m \leqslant \binom{a+b-t}{a-t}$.

9.3. Show that the upper bounds in Bollobás's and Füredi's theorems (Theorems 9.7 and 9.11) are tight.

> *Hint*: Take two disjoint sets X and S of respective sizes $a+b-2s$ and s. Arrange the s-element subsets of X in any order: $Y_1, Y_2, \ldots$. Let $A_i = S \cup Y_i$ and $B_i = S \cup (X - Y_i)$.

9.4. Use the binomial theorem to prove the following. Let $0 < p < 1$ be a real number, and $C \subset D$ be any two fixed subsets of $\{1, \ldots, n\}$. Then the sum of $p^{|A|}(1-p)^{n-|A|}$ over all sets A such that $C \subseteq A \subseteq D$, equals $p^{|C|}(1-p)^{n-|D|}$.