$$|A_1| + |A_k \cup \{x\}| = (|A_1| + |A_k|) + 1 = (n-1) + 1 = n$$

and

$$|A_1 \cup \{x\}| + |A_{k-1} \cup \{x\}| = (|A_1| + |A_{k-1}|) + 2 = (n-2) + 2 = n.$$

Is this a partition? It is indeed. If $A \subseteq Y$ then only $\mathcal{C}'_i$ contains A where $\mathcal{C}_i$ is the chain in 2^Y containing A. If $A = B \cup \{x\}$ where $B \subseteq Y$ then $B \in \mathcal{C}_i$ for some i. If B is the maximal element of $\mathcal{C}_i$ then $\mathcal{C}'_i$ is the only chain containing A, otherwise A is contained only in $\mathcal{C}''_i$. □

9.1.2 Application: the memory allocation problem

The following problem arises in information storage and retrieval. Suppose we have some list (a sequence) $L = (a_1, a_2, \ldots, a_m)$ of not necessarily distinct elements of some set X. We say that this list *contains* a subset A if it contains A as a subsequence of consecutive terms, that is, if

$$A = \{a_i, a_{i+1}, \ldots, a_{i+|A|-1}\}$$

for some i. A sequence is *universal* for X if it contains all the subsets of X. For example, if $X = \{1, 2, 3, 4, 5\}$ then the list

$$L = (1\,2\,3\,4\,5\,1\,2\,4\,1\,3\,5\,2\,4)$$

of length $m = 13$ is universal for X.

What is the length of a shortest universal sequence for an n-element set? Since any two sets of equal cardinality must start from different places of this string, the trivial lower bound for the length of universal sequence is $\binom{n}{\lfloor n/2 \rfloor}$, which is about $\sqrt{\frac{2}{\pi n}}2^n$, according to Stirling's formula (1.4). A trivial upper bound for the length of the shortest universal sequence is obtained by considering the sequence obtained simply by writing down each subset one after the other. Since there are 2^n subsets of average size $n/2$, the length of the resulting universal sequence is at most $n2^{n-1}$. Using Dilworth's theorem, we can obtain a universal sequence, which is n times (!) shorter than this trivial one.

Theorem 9.4 (Lipski 1978)**.** *There is a universal sequence for* $\{1, \ldots, n\}$ *of length at most* $\frac{2}{\pi}2^n$.

Proof. We consider the case when n is even, say $n = 2k$ (the case of odd n is similar). Let $S = \{1, \ldots, k\}$ be the set of the first k elements and $T = \{k+1, \ldots, 2k\}$ the set of the last k elements. By Theorem 9.3, both S and T have symmetric chain decompositions of their posets of subsets into $m = \binom{k}{k/2}$ symmetric chains: $2^S = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_m$ and $2^T = \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_m$. Corresponding to the chain

$$\mathcal{C}_i = \{x_1, \ldots, x_j\} \subset \{x_1, \ldots, x_j, x_{j+1}\} \subset \ldots \subset \{x_1, \ldots, x_h\} \quad (j + h = k)$$

we associate the sequence (not the set!) $C_i = (x_1, x_2, \ldots x_h)$. Then every subset of S occurs as an *initial* part of one of the sequences $C_1, \ldots, C_m$. Similarly let $D_1, \ldots, D_m$ be sequences corresponding to the chains $\mathcal{D}_1, \ldots, \mathcal{D}_m$. If we let $\overline{D}_i$ denote the sequence obtained by writing D_i in reverse order, then every subset of T occurs as a *final* part of one of the $\overline{D}_i$. Next, consider the sequence

$$L = \overline{D}_1 C_1 \overline{D}_1 C_2 \ldots \overline{D}_1 C_m \ldots \overline{D}_m C_1 \overline{D}_m C_2 \ldots \overline{D}_m C_m.$$

We claim that L is a universal sequence for the set $\{1, \ldots, n\}$. Indeed, each of its subsets A can be written as $A = E \cup F$ where $E \subseteq S$ and $F \subseteq T$. Now F occurs as the final part of some $\overline{D}_f$ and E occurs as the initial part of some C_e; hence, the whole set A occurs in the sequence L as the part of $\overline{D}_f C_e$. Thus, the sequence L contains every subset of $\{1, \ldots, n\}$. The length of the sequence L is at most $km^2 = k\binom{k}{k/2}^2$. Since, by Stirling's formula, $\binom{k}{k/2} \sim 2^k \sqrt{\frac{2}{k\pi}}$, the length of the sequence is $km^2 \sim k\frac{2}{k\pi} \cdot 2^{2k} = \frac{2}{\pi} 2^n$. □

9.2 Antichains

A set system $\mathcal{F}$ is an *antichain* (or *Sperner system*) if no set in it contains another: if $A, B \in \mathcal{F}$ and $A \neq B$ then $A \not\subseteq B$. It is an antichain in the sense that this property is the other extreme from that of the chain in which every pair of sets is comparable.

9.2.1 Sperner's theorem

Simplest examples of antichains over $\{1, \ldots, n\}$ are the families of all sets of fixed cardinality k, $k = 0, 1, \ldots, n$. Each of these antichains has $\binom{n}{k}$ members. Recognizing that the maximum of $\binom{n}{k}$ is achieved for $k = \lfloor n/2 \rfloor$, we conclude that there are antichains of size $\binom{n}{\lfloor n/2 \rfloor}$. Are these antichains the largest ones?

The positive answer to this question was found by Emanuel Sperner in 1928, and this result is known as Sperner's Theorem.

Theorem 9.5 (Sperner 1928)**.** *Let $\mathcal{F}$ be a family of subsets of an n element set. If $\mathcal{F}$ is an antichain then $|\mathcal{F}| \leqslant \binom{n}{\lfloor n/2 \rfloor}$.*

A considerably sharper result, Theorem 9.6 below, is due to Lubell (1966). The same result was discovered by Meshalkin (1963) and (not so explicitly) by Yamamoto (1954). Although Lubell's result is also a rather special case of an earlier result of Bollobás (see Theorem 9.8 below), inequality (9.1) has become known as the *LYM inequality*.

Theorem 9.6 (LYM Inequality)**.** *Let $\mathcal{F}$ be an antichain over a set X of n elements. Then*