1 More hints and solutions

1.16

We have

$$\left(\frac{n}{k(n-k)}\right)^{1/2} \geq \frac{1}{\sqrt{k}}.$$

Since $\ln(1+t) > t - \frac{1}{2}t^2$ for $t > 0$,

$$\begin{aligned} \ln\left(\frac{n}{n-k}\right)^{n-k} &> (n-k)\left[\frac{k}{n-k} - \frac{k^2}{2(n-k)^2}\right] \\ &\geq k - \frac{k^2}{n}. \end{aligned}$$

Hence $\gamma^{-1} = \sqrt{2\pi} \cdot e^{1/6k} \cdot e^{k^2/n}\sqrt{k} = \sqrt{2\pi k} \cdot e^{k^2/n+1/6k}$.

1.17

Say that a k-element subset $S \subseteq \{1, 2, \ldots, n\}$ is *good* if $x \neq y + 1$ for all $x, y \in S$, $x \neq y$. Our goal is to compute the number N of such subsets. Let $S = \{a_1, , a_2, \ldots, a_k\}$ be a good subset with $a_1 < a_2 < \ldots < a_k$. Then the set $S' = \{a_1, a_2 - 1, \ldots, a_k - (k-1)\}$ is a k-element subset of $\{1, 2, \ldots, n-k+1\}$. Hence, $L \leq \binom{n-k+1}{k}$. On the other hand, for every k-element subset $\{b_1 < b_2 < \ldots < b_k\}$ of $\{1, 2, \ldots, n-k+1\}$, the set $S = \{b_1, b_2 + 1, \ldots, b_k + (k-1)\}$ is a good subset of $\{1, 2, \ldots, n\}$. Hence, $L \geq \binom{n-k+1}{k}$.

1.26

Let n be the number of objects, z the number of bins, x the number of bins that are not red and y the number of bins that are not blue. There are z^n ways of sorting the objects into bins; x^n of these ways shun red and y^n of them shun blue. So where A is the number of ways that shun both colors, B is the number of ways that shun red but not blue, C is the number of ways that shun blue but not red, and D is the number of ways that shun neither, we have $x^n = A + B$, $y^n = A + C$ and $z^n = A + B + C + D$. These equations give $x^n + y^n = z^n$ if and only if $A = D$, which is to say, if and only if the number of ways of sorting objects into a row of colored bins that shun both colors is equal to the number of ways that shun neither.

2.8

By Eq. (1.8), the right-hand sum can be written as

$$\sum_{i_1=1}^{m} \cdots \sum_{i_{k-1}=1}^{m} \sum_{A_{i_k} \in \mathcal{F}} |Y \cap A_{i_k}| = \sum_{i_1=1}^{m} \cdots \sum_{i_{k-1}=1}^{m} \sum_{x \in Y} d(x),$$

where $Y = Y(i_1, \ldots, i_{k-1}) := A_{i_1} \cap \cdots \cap A_{i_{k-1}}$. Changing the order of the summation, we obtain $\sum_{x \in X} d(x) \cdot N(x)$, where $N(x)$ is the number of $(k-1)$-tuples of sets in $\mathcal{F}$ containing the point x. Since $N(x) = d(x)^{k-1}$, we are done.

2.15

First, by (1.7) and (1.9),

$$\sum_{i<j} |S_i \cap S_j| = \frac{1}{2}\left(\sum_{i,j} |S_i \cap S_j| - \sum_i |S_i|\right) = \frac{1}{2}\left(\sum_{x\in V} d(x)^2 - \sum_{x\in V} d(x)\right) = \sum_{x\in V}\binom{d(x)}{2}.$$

Now, by (ii),

$$\sum_{i<j} |S_i \cap S_j| = \sum_{\{i,j\}\in E} |S_i \cap S_j| \le k \cdot |E|.$$

Hence,

$$\sum_{x\in V}\binom{d(x)}{2} \le k \cdot |E|.$$

On the other hand, by (i) we know that

$$\sum_{x\in V} d(x) = \sum_{i=1}^{n} |S_i| \ge n \cdot r.$$

Hence, the sum $\sum_{x\in V}\binom{d(x)}{2}$ is minimized when $d(x) = r$ for all $x \in V$, implying that

$$n \cdot \binom{r}{2} = |V| \cdot \binom{r}{2} \le \sum_{x\in V}\binom{d(x)}{2} \le k \cdot |E|.$$

4.16

For an edge $e = \{x, y\}$, let $t(e)$ be the number of triangles containing e. Let $B = V \setminus \{x, y\}$. Among the vertices in B there are precisely $t(e)$ vertices which are adjacent to both x and y. Every other vertex in B is adjacent to at most one of these two vertices. Thus, $d(x) + d(y) - t(e) \le n$. Summing over all edges $e = \{x, y\}$ we obtain

$$\sum_{e\in E} (d(x) + d(y)) - \sum_{e\in E} t(e) \le n \cdot |E|.$$

The second term on the left-hand side is equal to $3 \cdot t(G)$ whereas the first is equal to $\sum_{x\in V} d(x)^2$ which, by Cauchy–Schwarz inequality is at least $\left(\sum_x d(x)\right)^2/n = 4 \cdot |E|^2/n$. Altogether this yields the desired lower bound on $t(G)$.

4.17

Let $G = (V, E)$ be a (k, r)-sparse graph on n vertices, and let N be the sum, over all k-element subsets S of V, of the number of edges spanned by S. That is, $N = \sum_S |E(S)|$ where $E(S)$ is the set of edges from E having both endpoints in S. Every edge of E is spanned by precisely $\binom{n-2}{k-2}$ of the sets S. By double-counting,

$$N = \sum_S \sum_{e\in E(S)} 1 = \sum_{e\in E} \sum_{S: e\in E(S)} 1 = |E| \cdot \binom{n-2}{k-2}.$$

Hence

$$\frac{|E|}{\binom{n}{2}} = \frac{|E| \cdot \binom{n-2}{k-2}}{\binom{k}{2} \cdot \binom{n}{k}} = \frac{N}{\binom{k}{2} \cdot \binom{n}{k}} \leq \frac{r}{\binom{k}{2}};$$

here the first equality is Exercise 1.10, and the inequality holds because $N/\binom{n}{k}$ is the average number of edges in E spanned by a k-element set, and hence, cannot exceed r.

9.4

Observe that

$$\sum_{C \subseteq A \subseteq D} p^{|A|} q^{n-|A|} = \sum_{B \subseteq D \setminus C} p^{|B|+|C|} q^{n-|B|-|C|}$$

and that by the binomial theorem, for any set X we have that

$$\sum_{B \subseteq X} p^{|B|} q^{|X|-|B|} = \sum_{i=0}^{|X|} \binom{|X|}{i} p^i q^{|X|-i} = (p+q)^{|X|} = 1.$$

13.7

Suppose not, i.e., $L \cap S = \{x\}$ for some $x \in S$ and line L. There are $q+1$ points in S besides x, and for every such points there must be a line $\neq L$, connecting it with x. So, we would have $q+2$ lines through x, a contradiction.

13.12

$\sum_{x \in F_p} x^t = \sum_{i=0}^{p-2} (a^i)^t = \sum_{i=0}^{p-2} (a^t)^i = ((a^t)^{p-1} - 1)/(a^t - 1).$

20.16

Let I_j, $j = 1, \ldots, n$, be the set of positions in w where the jth letter of the alphabet appears. Assume that $|I_i| = N/n$ for all i. The number of good r-subsets $w_{i_1}, \ldots, w_{i_r}$ equals the number $\binom{n}{r}$ of possibilities to choose r blocks of positions, times the number $(N/n)^r$ of possibilities to arrange the corresponding (to these blocks) r letters to their positions (one letter can appear in N/n possible positions). Thus

$$\mu(w,r) \leq \frac{\left(\frac{N}{n}\right)^r \cdot \binom{n}{r}}{\binom{N}{r}} \leq \frac{\left(\frac{N}{n}\right)^r \cdot \binom{n}{r}}{\left(\frac{N}{r}\right)^r} = \frac{(n)_r}{n^r} \cdot \frac{r^r}{r!} \sim \frac{(n)_r}{n^r} \cdot e^r.$$

To get rid of the e^r term, use more tight lower bound for the binomial coefficient, given in Ex. 1.16.

2 More exercises

2.1 A Ramsey-type theorem for set intersections

Use double-counting to prove the following Ramsey-type result:

Given any integer $k \geq 2$, there exists an integer n such that given any n n-elements subsets of $[2n-1] = \{1, 2, \ldots, 2n-1\}$, there exist k of these subsets with at least k elements in common.

Show that the result holds for $n \leq k \cdot 2^k + \binom{k}{2} + 1$.

Hint: (Ramras 2002): For each $n > k$ consider bipartite graphs $G = (X, Y, E)$ where $X =$ any family of n n-elements subsets of $[2n-1]$; $Y =$ the family of all k-element subsets of $[2n-1]$, and $(x, y) \in E$ iff $x \supset y$. Suppose the statement is false, and show that then $\deg(y) \leq k-1$ for every $y \in Y$. Use this to get a contradiction with the (obvious) fact that $\sum_{x \in X} \deg(x) = \sum_{y \in Y} \deg(y)$.

To get the desired upper bound on n, observe that

$$n\binom{n}{k} = |E| \leq \binom{2n-1}{k} \cdot \max_{y \in Y} \deg(y).$$

2.2 Ramsey theorem for bipartite graphs

Use the Pigeon-hole principle and the previous exercise to prove the following:

For any integer k there exists an integer m such that given any r-coloring of the edges of the complete bipartite graph $K_{m,m}$ there exists a monochromatic induced subgraph isomorphic to $K_{k,k}$.

Hint: Choose n according to the previous exercise, and let $m = nr - 1$. Let the bipartition of $K_{m,m}$ be (A, B). Apply the Pigeonhole principle to show that, for each vertex $a \in A$, some n edges incident with a must receive the same color; call that color $c(a)$, and assign vertex a the color $c(a)$. Apply the Pigeonhole principle once again, this time to the r-colored vertices of A, to show that some set $A' \subseteq A$ of $|A'| = n$ vertices of A will receive the same color, say, red color. Consider the family of n sets $R(a) = \{b \in B : \text{edge } (a, b) \text{ is red}\}$ with $a \in A'$, and apply the previous exercise.

2.3 List chromatic number of bipartite graphs

Let $G = (V_1, V_2, E)$ be a bipartite graph with $|V_1| = |V_2| = n$ and with a list C_v of at least $\log_2 n$ colors associated with each vertex v. Prove that there is a legal coloring of G assigning to each vertex v a color from its list C_v.

Comment: Hence, every bipartite $n \times n$ graph G has list chromatic number

$$\chi_\ell(G) \leq \log_2 n.$$

Hint: Let C be the union of all sets C_v. For each $c \in C$ choose, randomly and independently, a value $i_c \in \{1, 2\}$. The colors c for which $i_c = i$ will be the ones to be used for coloring the vertices in V_i. Use the counting sieve to prove that for every $i \in \{1, 2\}$ and for every $v \in V_i$, there is at least one color $c \in C_v$ such that $i_c = i$.

2.4 Rich submatrices

Let M be a matrix with arbitrary entries. Let $\Delta(M)$ be the minimum number such that in every row and in every column each entry can appear at most $\Delta(M)$ times. Prove the following:

In every $t^2 \times t^2$ matrix M there is a $t \times t$ submatrix containing at least

$$\frac{t^2}{4\Delta(M)}$$

different entries.

Sketch: (Ajtai 1999) Let $M(X,Y) = \{M(x,y) : x \in X, y \in Y\}$ be an arbitrary matrix with $|X| = |Y| = t^2$. Apply the following greedy strategy to construct the sequence of pairs of sets of rows $X_i = \{x_1, \ldots, x_i\}$ and columns $Y_i = \{y_1, \ldots, y_i\}$ for $i = 1, \ldots, t$: at i-th step pick $x_i \in X$ and $y_i \in Y$ so that the difference $D_i = |M(X_i, Y_i)| - |M(X_{i-1}, Y_{i-1})|$ is maximal. It is enough to show that, for every $i > \frac{t}{2} + 1$, either $|M(X_{i-1}, Y_{i-1})| \geq \frac{t^2}{4\Delta}$ (and we can stop the procedure) or $D_i \geq \frac{3t}{8\Delta}$. For this, assume that $i > \frac{t}{2} + 1$ but $|M(X_{i-1}, Y_{i-1})| < \frac{t^2}{4\Delta}$. Argue that then, for every $j = 1, \ldots, i-1$, the set

$$W_j = \{y \in Y : M(x_j, y) \notin M(X_{i-1}, Y_{i-1})\}$$

has at least $\frac{3t^2}{4\Delta}$ elements. Hence, $\sum_{j=1}^{i-1} |W_j| \geq \frac{3t^3}{8\Delta}$. Use double-counting to show that some $y \in Y \setminus Y_{i-1}$ must belong to at least $\frac{3t}{8\Delta}$ sets W_j. Take $y_i = y$ (why $y \notin Y_{i-1}$?) and pick an $x_i \in X \setminus X_{i-1}$ arbitrarily to show that $D_i \geq \frac{3t}{8\Delta}$.

2.5 Degree of induced subgraphs

For a graph $G = (V, E)$, let $d_{\text{ave}}(G) = \frac{1}{|V|} \sum_{v \in V} d(v)$ be the average degree and $d_{\min}(G) = \min\{d(v) : v \in V\}$ the minimum degree of its vertices. Prove the following: every graph G contains an induced subgraph H such that $d_{\text{ave}}(H) \geq d_{\text{ave}}(G)$ and $d_{\min}(H) \geq \frac{1}{2} d_{\text{ave}}(H)$.

Hint: Try to delete vertices of small degree one by one, until only vertices of large degree remain.

2.6 Clique number of 4-cycle-free graphs

A graph is C_4-*free* if it contains no cycle of length four as an induced(!) subgraph. Recall that the clique number $\omega(G)$ (resp., the independence number $\alpha(G)$) of a graph G denotes the maximum number of vertices of G all (resp., none) of which are adjacent.

(a) Show that for every C_4-free graph $G = (V, E)$,

$$\omega(G) \geq \frac{|V|}{\binom{\alpha(G)+1}{2}}.$$

Hint: Fix an independent set $S = \{x_1, \ldots, x_\alpha\}$ with $\alpha = \alpha(G)$. Let A_i be the set of neighbors of x_i in G, and B_i the set of vertices whose only neighbor in S is x_i. Consider the family $\mathcal{F}$ consisting of all α sets $\{x_i\} \cup B_i$ and $\binom{\alpha}{2}$ sets $A_i \cap A_j$. Show that: (i) each member of $\mathcal{F}$ forms a clique in G, and (ii) the members of $\mathcal{F}$ cover all vertices of G.

(b) Let G be a C_4-free graph with n vertices and minimum degree d. Prove that for every $t \leq \alpha(G)$,

$$\omega(G) \geq \frac{d \cdot t - n}{\binom{t}{2}}.$$

Hint: Take an independent set $S = \{x_1, \ldots, x_t\}$ of size t and let A_i be the set of neighbors of x_i in G. Let m be the maximum of $|A_i \cap A_j|$ over all $1 \leq i < j \leq t$. Use the inclusion-exclusion principle to show that

$$\left| \bigcup_{i=1}^{t} A_i \right| \geq td - \binom{t}{2} m$$

and argue that $m \leq \omega(G)$.

(c) Combine parts (a) and (b) to prove the following result due to Gyárfás, Hubenko and Solymosi (*Combinatorica*, **22**:2, 2002): there is an absolute constant $c > 0$ such that if $G = (V, E)$ is a C_4-free graph on $|V| = n$ vertices, then

$$\omega(G) \geq \frac{c|E|^2}{n^3}.$$

Comment: Note that being C_4-free here is very important: for example, a complete bipartite graph $K_{n,n}$ has $n^2/4$ edges but $\omega(K_{n,n}) = 2$.

Hint: Let a be the average degree of G; hence, $a = 2|E|/n$. By the previous exercise (about degrees of induced subgraphs), we already know that G has an induced subgraph of average degree $\geq a$ and minimum degree $\geq a/2$. So, we may assume w.l.o.g. (why?) that the graph G itself has these two properties. Now consider two cases depending on whether $\alpha(G)$ is larger than Cn/a or not (for an appropriately chosen constant C). If yes, apply part (b); if not, apply part (a). Show that in both cases, $\omega(G) = \Omega(a^2/n)$.

2.7 Zero-patterns of polynomials

Let $\mathbf{f} = \{f_i(x_1, \ldots, x_n) : i = 1, \ldots, m\}$ be a sequence of polynomials over some field F. For $v \in F^n$, a *zero-pattern* of $\mathbf{f}$ on a v is the set

$$S(\mathbf{f}, v) = \{i : f_i(v) = 0\} \subseteq \{1, \ldots, m\};$$

the point v is a *witness* for this zero-pattern. Let $Z_F(\mathbf{f})$ denote the number of zero-patterns of $\mathbf{f}$ as v ranges over F^n. Let d_i denote the degree of f_i, and $D = \sum_{i=1}^{m} d_i$. Prove that

$$Z_F(\mathbf{f}) \leq \binom{n+D}{n}.$$

Sketch: (Rónyai, Babai, and Ganapathy, *J. of AMS*, 14:3, 2001) Assume that $\mathbf{f}$ has M zero-patterns, and let $v_1, \dots, v_M$ be witnesses to each zero-pattern. Let $S_i = S(\mathbf{f}, v_i)$ and consider the polynomials $g_i = \prod_{k \in S_i} f_k$. Observe that $g_i(v_j) \neq 0$ if and only if $S_i \subseteq S_j$, and show that the polynomials $g_1, \dots, g_M$ are linearly independent over F.

2.8 Matrix rank and Ramsey graphs

Let R be a ring and $A = (a_{ij})$ be an $n \times n$ matrix with entries from R. The rank $\mathrm{rk}_R(A)$ of A over R is defined as the minimum number r for which there exists an $n \times r$ matrix B and an $r \times n$ matrix C over R such that $A = B \cdot C$; if all entries of A are zeroes then $\mathrm{rk}_R(A) = 0$.

(a) Suppose that $R = F$ is a field. Show that then $\mathrm{rk}_R(A)$ coincides with the usual matrix rank over F.

Hint: $B \cdot C$ is a set of linear combinations of the rows of B, given by columns of C.

(b) Suppose that A has no zero column and that every row of A contains at most s non-zero entries. Prove that $\mathrm{rk}_R(A) \geq n/s$.

(c) A matrix $A = (a_{ij})$ is (lower) *co-triangular* if $a_{ii} = 0$ and $a_{ij} \neq 0$ for all $1 \leq j < i \leq n$. Prove that if $R = GF(p)$ for some prime p, then $\mathrm{rk}_R(A) \geq n^{1/(p-1)} - p$.

Hint: Since p is a prime, $a^{p-1} = 1$ for every $a \neq 0$ in $GF(p)$. Use part (a) to represent the matrix as the product $A = B \cdot C$ of two matrices. For $i = 1, \dots, n$ consider the polynomials $f_i(x) = 1 - g_i(x)^{p-1}$ in r variables $x = (x_1, \dots, x_r)$ over $GF(p)$, where $g_i(x)$ is the scalar product of x with the i-th row of B, and show that the polynomials $f_1, \dots, f_n$ are linearly independent over $GF(p)$.

(d) (Grolmusz 2000) Consider the ring Z_6 of integers modulo 6, and let $A = (a_{ij})$ be an $n \times n$ co-triangular matrix over Z_6. Consider the graph $G_A = (V, E)$ with $V = \{1, \dots, n\}$; two vertices i and j are adjacent iff $i > j$ and a_{ij} is odd. Prove the following: if $r = \mathrm{rk}_{Z_6}(A)$ then the graph G_A contains neither a clique on $r+2$ vertices nor an independent set of size $(r+3)^2 + 1$.

Comment: Hence, low-rank co-triangular matrices can be used to construct graphs with good Ramsey properties.

Hint: It is clear that that $\mathrm{rk}_{GF(p)}(A) \leq r$ for both $p = 2, 3$. Show that every clique in G_A of size t corresponds to a $t \times t$ lower co-triangular submatrix of A over $GF(2)$, and every independent set of site t corresponds to a $t \times t$ lower co-triangular submatrix of A over $GF(3)$. In both cases apply the estimate $t \leq (r+p)^{p-1}$ from part (c).

2.9 Rank of generalized intersection matrices

Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a family of subsets of $[n] = \{1, \ldots, n\}$, and

$$f(x_1, \ldots, x_n) = \sum_{I \subseteq [n]} a_I X_I$$

be a multi-linear polynomial, where $X_I = \prod_{i \in I} x_i$. Assume that all the coefficients a_I either all are non-negative integers, or all belong to the ring Z_z for some r. The *weight* of f is the number $w(f)$ of monomials in f with nonzero coefficients. Let $f(\mathcal{A}) = \{B_1, \ldots, B_m\}$ denote the family of subsets of monomials(!) of f which is defined as follows. Take the incidence $n \times m$ matrix M of $\mathcal{A}$. The rows of the incidence matrix N of $f(\mathcal{A})$ correspond to monomials of f; there are a_I identical rows of N corresponding to the same monomial X_I. The row corresponding to a monomial $X_I = \prod_{i \in I} x_i$ of f is just a component-wise AND of the rows i of M with $i \in I$.

(a) Show that

$$f(A_i \cap A_j) = |B_i \cap B_j|$$

for all i, j; here $f(A_i \cap A_j)$ denotes the value of f on the incidence vector of $A_i \cap A_j$.

(b) Show that the matrix

$$I_f(\mathcal{A}) = \{f(A_i \cap A_j) : 1 \leq i, j \leq m\}$$

has rank at most $w(f)$ over the field of real numbers.

Hint: Use part (a) of the previous exercise to show that if M is a square matrix of intersection sizes of subsets of some domain D, then $\mathrm{rk}_R(M) \leq |D|$, and apply part (a) of this exercise.

2.10 Zeroes of multivariate polynomials

Let $f(x_1, \ldots, x_n)$ be a polynomial in n variables over an arbitrary field F. Suppose that the degree of f as a polynomial in x_i is d_i, for $1 \leq i \leq n$. Let $S_i \subset F$ be a set of at least $d_i + 1$ distinct elements of F, $i = 1, \ldots, n$. Prove the following:

If f is not the zero polynomial, then $f(s_1, \ldots, s_n) \neq 0$ for at least one point $(s_1, \ldots, s_n)$ in $S_1 \times \cdots \times S_n$.

Comment: Note that this is a “granulated”version of Zippel’s lemma (Lemma 25.2 in the book) where all sets S_i are required to have the size

$$|S_i| \geq \max\{d_1, \ldots, d_n\} + 1.$$

Hint: (Alon and Tarsi 1992) Prove the reversed claim: if $f(s_1, \ldots, s_n) = 0$ for all n-tuples $(s_1, \ldots, s_n)$ in $S_1 \times \cdots \times S_n$, then $f \equiv 0$. Proceed by induction on n. In the induction step, write f as a polynomial in x_n, that is

$$f = \sum_{i=1}^{d_n} f_i(x_1, \ldots, x_{n-1}) x_n^i$$

Show that all the polynomials f_i vanish on $S_1 \times \cdots \times S_{n-1}$, and apply the induction hypothesis.

2.11 Nullstellensatz

Let $f(x_1, \ldots, x_n)$ be a polynomial in n variables over an arbitrary field F, and let $\deg(f)$ denote the total degree of f. Prove the following special case of Hilbert's Nullstellensatz:

Let $S_1, \ldots, S_n$ be nonempty subsets of F and define

$$g_i(x_i) = \prod_{s \in S_i} (x_i - s).$$

If $f(s_1, \ldots, s_n) = 0$ for all $(s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$ (that is, if f vanishes on all the common zeros of $g_1, \ldots, g_n$), then there are polynomials $h_i(x_1, \ldots, x_n)$, $i = 1, \ldots, n$ satisfying $\deg(h_i) \leq \deg(f) - \deg(g_i)$ so that

$$f = \sum_{i=1}^{n} h_i g_i.$$

Hint: (Noga Alon) Define $d_i = |S_i| - 1$ for all i, and consider polynomials

$$g_i(x_i) = \prod_{s \in S_i} (x_i - s) = x_i^{d_i+1} - \sum_{j=0}^{d_i} a_{ij} x_i^j.$$

Observe that if $x_i \in S_i$ then $g_i(x_i) = 0$, that is,

$$x_i^{d_i+1} = \sum_{j=0}^{d_i} a_{ij} x_i^j. \qquad (*)$$

Let f' be the polynomial obtained by writing f as a linear combination of monomials and replacing, repeatedly, each occurrence of $x_i^{f_i}$ $(1 \leq i \leq n)$, where $f_i > d_i$, by a linear combination of smaller powers of x_i, using the relations $(*)$. Show that: (i) f' is obtained from f by subtracting from it products of the form $h_i g_i$ where $\deg(h_i) \leq \deg(f) - \deg(g_i)$; (ii) $f'(x) = f(x)$ for all $x \in S_1 \times \cdots \times S_n$, and (iii) $f'(x) = 0$ for all $x \in S_1 \times \cdots \times S_n$, and use the previous exercise.

2.12 Combinatorial Nullstellensatz

Let $f(x_1, \ldots, x_n)$ be a polynomial in n variables over an arbitrary field F. Prove the following:

Let $t_1, \ldots, t_n$ be no-negative integers such that $\sum_{i=1}^{n} t_i = \deg(f)$. If the coefficient of the monomial $\prod_{i=1}^{n} x_i^{t_i}$ in f is nonzero, then for any subsets $S_1, \ldots, S_n$ of F of size $|S_i| \geq t_i + 1$, there exists a point $(a_1, \ldots, a_n)$ in $S_1 \times \cdots \times S_n$ for which $f(a_1, \ldots, a_n) \neq 0$.

Hint: (Noga Alon) We may assume that $|S_i| = t_i + 1$ for all i. Suppose the result if false, and define $g_i(x_i) = \prod_{s \in S_i} (x_i - s)$. Let $h_1, \ldots, h_n$ be the polynomials guaranteed by the Nullstellensatz from the previous exercise. Let a and b be, respectively, the coefficients of $\prod_{i=1}^{n} x_i^{t_i}$ in f and in $\sum_{i=1}^{n} h_i g_i$. By assumption, $a \neq 0$, and hence, it should be that $b \neq 0$. Use the facts that $\deg(h_i g_i) \leq \deg(f)$ and that $x_i^{t_i+1}$ is a monomial of $g_i(x_i)$ to show that $b = 0$.

2.13 Regular subgraphs

A graph is *p-regular* if all its vertices have degree p. Use the abridged version of Combinatorial Nullstellensatz to prove the following theorem of Alon, Friedland and Kalai (*J. Comb. Theor.* **B 37**, 1984):

> Let $G = (V, E)$ be a graph. Assume that G has no loops but multiple edges are allowed. Let p be a prime number. If G has average degree bigger that $2p - 2$ and maximum degree at most $2p - 1$, then G contains a spanning p-regular subgraph.

> *Sketch*: (Noga Alon) Associate each edge e of G with a variable x_e and consider the polynomial
>
> $$f = \prod_{v \in V} \left[1 - \left(\sum_{e \in E} a_{v,e} x_e \right)^{p-1} \right] - \prod_{e \in E} (1 - x_e)$$
>
> over $GF(p)$, where $a_{v,e} = 1$ if $v \in e$ and $a_{v,e} = 0$ otherwise. Show that the total degree of f is $|E|$ and apply the abridged version of the Combinatorial Nullstellensatz to f. Take a point $x = (x_e : e \in E)$ for which $f(x) \neq 0$. Argue that $x \neq (0, \dots, 0)$ and that, for this vector, $\sum_{e \in E} a_{v,e} x_e$ is divisible by p for every v. Take the subgraph H consisting of all edges $e \in E$ for which $x_e = 1$, and show that this subgraph must be p-regular.

2.14 The permanent lemma

Let $A = (a_{ij})$ be an $n \times n$ matrix over a field F. The *permanent* $\mathrm{Per}(A)$ of A is a sum of $n!$ products $a_{1i_1} a_{2i_2} \cdots a_{ni_n}$, where $(i_1, i_2, \dots, i_n)$ is a permutation of $(1, 2, \dots, n)$. Prove the following:

> If $\mathrm{Per}(A) \neq 0$, then for any vector $b \in F^n$, there is a subset of the rows of A whose sum differs from b in all coordinates.

> *Hint*: Consider the polynomial $f = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j - b_j \right)$ and apply the abridged version of Combinatorial Nullstellensatz with all $S_i = \{0, 1\}$.