Lecture Notes in

GRAPH THEORY

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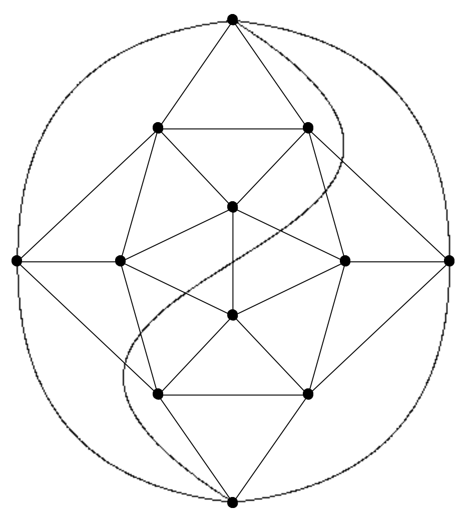
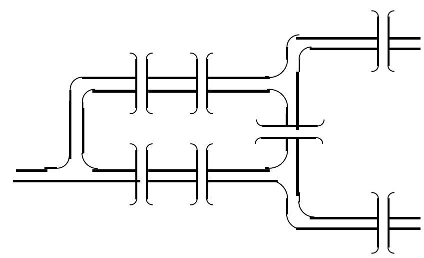


Preface

Graph theory can be said to have its beginning in 1736 when EULER considered the (general case of the) *Königsberg bridge problem*: Is there a walking route that crosses each of the seven bridges of Königsberg exactly once?



It took 200 years before the first book on graph theory was written. This was done by KÖNIG in 1936. Since then graph theory has developed into an extensive and popular branch of mathematics, which has been applied to many problems in mathematics, computer science, and other scientific and not-so-scientific areas. For the history of graph theory, see

N.L. BIGGS, R.J. LLOYD AND R.J. WILSON, "Graph Theory 1736 – 1936", Clarendon Press, 1986.

There seem to be no standard notations or even definitions for graph theoretical objects. This is natural, because the names one uses for these objects reflect the applications. So, for instance, if we consider a communications network (say, for e-mail) as a graph, then the computers, which take part in this network, are called nodes rather than vertices or points. On the other hand, other names are used for molecular structures in chemistry, flow charts in programming, human relations in social sciences, and so on.

These lectures study *finite graphs* and majority of the topics is included in

J.A. BONDY AND U.S.R. MURTY, "Graph Theory with Applications", Macmillan, 1978.

R. DIESTEL, "Graph Theory", Springer-Verlag, 1997.

F. HARARY, "Graph Theory", Addison-Wesley, 1969.

D.B. WEST, "Introduction to Graph Theory", Prentice Hall, 1996.

R.J. WILSON, "Introduction to Graph Theory", Longman, (3rd ed.) 1985.

In these lectures we study *combinatorial aspects* of graphs. For more algebraic topics and methods one should consult

N. BIGGS, "Algebraic Graph Theory", Cambridge University Press, (2nd ed.) 1993.

Notations and notions

- For a finite set X, we let $|X|$ denote its size (cardinality), that is, the number of its elements.
- Let $[1,n] = \{1,2,\dots,n\}$, and generally, $[i,n] = \{i, i+1, \dots, n\}$ for integers $i \leq n$.
- A family $\{X_1, X_2, \dots, X_k\}$ of subsets $X_i \subseteq X$ of a set X is a **partition** of X, if

$$X = \bigcup_{i \in [1,k]} X_i \quad \text{and} \quad X_i \cap X_j = \emptyset \text{ for all different } i \text{ and } j\,.$$

- For two sets X and Y,

$$X \times Y = \{(x,y) \mid x \in X,\ y \in Y\}$$

  is their **Cartesian product**.
- Two numbers $n, k \in \mathbb{N}$ (often $n = |X|$ and $k = |Y|$ for sets X and Y) have the **same parity**, if both are even, or both are odd, that is, if $n \equiv k \pmod 2$. Otherwise, they have opposite parity.

Graph theory has abundant examples of **NP-complete problems**. Intuitively, a problem is in P [1] if there is an efficient (practical) algorithm to find a solution to it. On the other hand, a problem is in NP [2], if it is first efficient to guess a solution and then efficient to check that this solution is correct. It is conjectured (and not known) that P $\neq$ NP. This is one of the great problems in modern mathematics and theoretical computer science. If the guessing in NP-problems can be replaced by an efficient systematic search for a solution, then P=NP. For any one NP-complete problem, if it is in P, then necessarily P=NP.

Sections with a star ($*$) in their heading are optional.

[1] Solvable – by an algorithm – in polynomially many steps on the size of the problem instances.
[2] Solvable *nondeterministically* in polynomially many steps on the size of the problem instances.

1 Basics of Graph Theory

Graphs and their plane figures

Let V be a *finite* set, and denote by

$$E(V) = \{\{u, v\} \mid u, v \in V,\ u \neq v\}\,.$$

the subsets of V of two distinct elements.

DEFINITION. A pair $G = (V, E)$ with $E \subseteq E(V)$ is called a **graph** (**on** V). The elements of V are the **vertices**, and those of E the **edges** of the graph. □

In literature, graphs are also called *simple graphs*; vertices are called *nodes* or *points*; edges are called *lines* or *links*. The list of alternatives is long (but still finite).

We adopt some notational conventions. The vertex set of a graph G is denoted by V_G and its edge set by E_G. Therefore $G = (V_G, E_G)$. A pair $\{u, v\}$ is usually written simply as uv. Notice that then $uv = vu$.

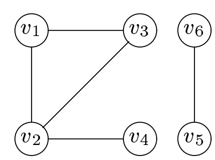
DEFINITION. For a graph G, we let

$$|V_G| = \nu_G\ , \qquad |E_G| = \varepsilon_G\,.$$

The number ν_G of the vertices is called the **order** of G.

For an edge $e = uv \in E_G$, the vertices u and v are its **ends**. Vertices u and v are **adjacent**, or **neighbours**, if $e = uv \in E_G$. Two edges $e_1 = uv$ and $e_2 = uw$ having a common end, are **adjacent** with each other. □

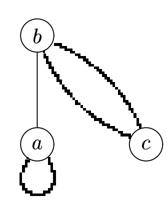
A graph G can be represented as a plane figure by drawing a line (or a curve) between the points u and v (representing vertices) if $e = uv$ is an edge of G. The figure on the right is a drawing of the graph G with $V_G = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E_G = \{v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_5v_6\}$.



Often we shall omit the identities (names v) of the vertices in our figures, in which case the vertices are drawn as anonymous circles.

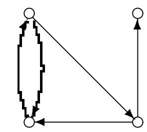
Graphs can be generalized by allowing **loops** vv and **parallel** (or **multiple**) **edges** between vertices to obtain a **multigraph** $G = (V, E, \psi)$, where $E = \{e_1, e_2, \dots, e_m\}$ is a set (of symbols), and $\psi\colon E \to E(V) \cup \{vv \mid v \in V\}$ is a function that attaches an unordered pair of vertices to each $e \in E$: $\psi(e) = uv$.

Note that we can have $\psi(e_1) = \psi(e_2)$. This is drawn in the figure of G by placing two (parallel) edges that connect the common ends. On the right there is (a drawing of) a multigraph G with vertices $V = \{a, b, c\}$ and edges $\psi(e_1) = aa$, $\psi(e_2) = ab$, $\psi(e_3) = bc$, and $\psi(e_4) = bc$.



Later we concentrate on (simple) graphs.

DEFINITION. We also study **directed graphs** or **digraphs** $D = (V, E)$, where the edges have a direction, that is, the edges are ordered: $E \subseteq V \times V$. In this case, $uv \neq vu$. □



The directed graphs have representations, where the edges are drawn as arrows. A digraph can contain edges uv and vu of *opposite directions.*

Graphs and digraphs can also be coloured, labelled, and weighted:

DEFINITION. A function $\alpha\colon V_G \to K$ is a **vertex colouring** of G by a set K of colours. A function $\alpha\colon E_G \to K$ is an **edge colouring** of G. Usually, $K = [1, k]$.

If $K \subseteq \mathbb{R}$ (often $\mathbb{N}$), then α is a **weight function** or a **distance function**. □

Isomorphism

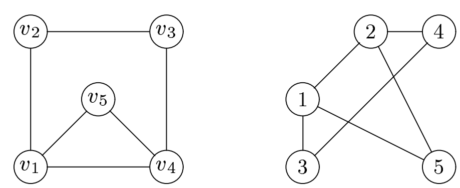
DEFINITION. Two graphs G and H are **isomorphic**, denoted by $G \cong H$, if there exists a bijection $\alpha\colon V_G \to V_H$ such that

$$uv \in E_G \iff \alpha(u)\alpha(v) \in E_H$$

for all $u, v \in V_G$. □

Hence G and H are isomorphic if the vertices of H are renamings of those of G. Two isomorphic graphs enjoy the same graph theoretical properties, and *they are often identified.* In particular, all isomorphic graphs have the same plane figures (excepting the identities of the vertices). This shows in the figures, where we tend to replace the vertices by small circles, and talk of 'the graph' although there are, in fact, infinitely many of such graphs.

The following graphs are isomorphic.



Research problem. *Does there exist an efficient algorithm to check whether any two given graphs are isomorphic or not?*

The following table lists the number $2^{\binom{n}{2}}$ of graphs on a given set of n vertices, and the number of nonisomorphic graphs on n vertices. It tells that at least for computational purposes an efficient algorithm for checking whether two graphs are isomorphic or not would be greatly appreciated.

n	1	2	3	4	5	6	7	8	9
graphs	1	2	8	64	1024	32 768	2 097 152	268 435 456	$2^{36} > 6 \cdot 10^{10}$
nonisomorphic	1	2	4	11	34	156	1044	12 346	274 668

Other representations

Plane figures catch graphs for our eyes, but if a problem on graphs is to be *programmed*, then these figures are (to say the least) unsuitable. Matrices of integers are ideal for computers, since every respectable programming language has array structures for these, and computers are good in crunching numbers.

Let $V_G = \{v_1, \ldots, v_n\}$ be ordered. The **adjacency matrix** of G is the $n \times n$-matrix M with entries $M_{ij} = 1$ or $M_{ij} = 0$ according to whether $v_i v_j \in E_G$ or not. As an example, the previous graph G has an adjacency matrix on the right. Notice that the adjacency matrix is always symmetric (with respect to its diagonal).

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The following result is obvious.

Theorem 1.1. *Two graphs G and H are isomorphic if and only if they have a common adjacency matrix.*

Graphs can also be represented by sets. For this, let $\mathcal{X} = \{X_1, X_2, \ldots, X_n\}$ be a family of subsets of a set X, and define the **intersection graph** $G_{\mathcal{X}}$ as the graph with vertices $X_1, \ldots, X_n$, and edges $X_i X_j$ for all i and j $(i \neq j)$ with $X_i \cap X_j \neq \emptyset$.

As an exercise we state:

Theorem 1.2. *Every graph is an intersection graph of some family of subsets.*

Let $s(G) = \min\{|X| \mid G \cong G_{\mathcal{X}} \text{ for some } \mathcal{X} \subseteq 2^X\}$. How small can $s(G)$ be compared to ν_G (or ε_G)? It was shown by Kou, Stockmeyer and Wong (1976) that to determine $s(G)$ is algorithmically difficult – the problem is NP-complete.

As another example, let A be a finite set of natural numbers, and let $G_A = (A, E)$ be a graph defined on A such that $rs \in E$ if and only if r and s (for $r \neq s$) have a common divisor > 1. All graphs can be represented in this form.

Degrees

Definition. Let $v \in V_G$ be a vertex a graph G. The **neighbourhood** of v is the set

$$N_G(v) = \{u \in V_G \mid vu \in E_G\}.$$

The **degree** of v is the number of its neighbours:

$$d_G(v) = |N_G(v)|.$$

If $d_G(v) = 0$, then v is **isolated**, and if $d_G(v) = 1$, then v is a **leaf**.

The **minimum degree** and the **maximum degree** of G are defined as

$$\delta(G) = \min\{d_G(v) \mid v \in V_G\} \quad \text{and} \quad \Delta(G) = \max\{d_G(v) \mid v \in V_G\}. \qquad \square$$

The next **handshaking lemma** tells that if several people shake hands, then the number of hands shaken is even. Indeed, we need two hands in each handshake.

Lemma 1.3. *For each graph G,*

$$\sum_{v \in V_G} d_G(v) = 2 \cdot \varepsilon_G .$$

Moreover, the number of vertices of odd degree is even.

Proof. Every edge $e \in E_G$ has two ends. The second claim follows immediately from the first one. □

Lemma 1.3 holds equally well for multigraphs, when $d_G(v)$ is defined as the number of edges that have v as an end, and when *a loop vv is counted twice.*

Special graphs

DEFINITION. A graph $G = (V, E)$ is **trivial**, if $\nu_G = 1$; otherwise G is **nontrivial**.

The graph $G = K_V$ is the **clique** (or the *complete graph*) on V, if every two vertices are adjacent: $E = E(V)$. All cliques of order n are isomorphic with each other, and they will be denoted by K_n.

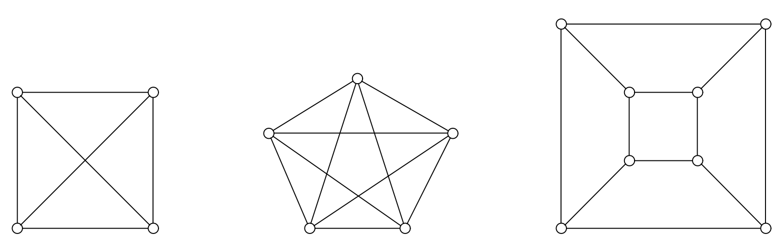
The **complement** of G is the graph $\overline{G}$ on V_G, where

$$E_{\overline{G}} = \{e \in E(V) \mid e \notin E_G\} .$$

The graph $G = \overline{K}_V$ is a **discrete graph**, when $E_G = \emptyset$. Clearly, all discrete graphs of order n are isomorphic with each other.

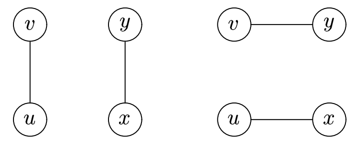
The graph G is **regular**, if every vertex has the same degree. If this degree is equal to r, then G is r-**regular** or **regular of degree** r. □

Note that a discrete graph is 0-regular, and a clique K_n is $(n-1)$-regular.



K_4 K_5 3-regular

Example 1.4. Let G be a graph. A **2-switch** $(u, v; x, y)$ of G, for $uv, xy \in E_G$ and $ux, vy \notin E_G$, replaces the edges uv and xy by ux and vy. Berge (1973) showed:



Two graphs G and H on a common vertex set V satisfy $d_G(v) = d_H(v)$ for all $v \in V$ if and only if H can be obtained from G by a sequence of 2-switches. □

Subgraphs

Definition. A graph H is a **subgraph** of a graph G, denoted by $H \subseteq G$, if $V_H \subseteq V_G$ and $E_H \subseteq E_G$.

A subgraph H **spans** G (and H is a **spanning subgraph** of G), if $V_H = V_G$.

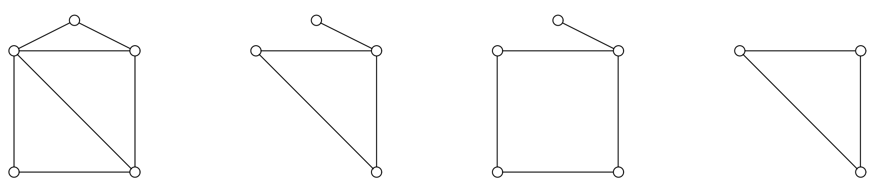
A subgraph H is an **induced subgraph** of G, if $E_H = E_G \cap E(V_H)$. In this case, H is **induced** by its set V_H of vertices. □

In an induced subgraph H of G, the set E_H of edges consists of all $e \in E_G$ such that $e \in E(V_H)$. Now to every nonempty subset $A \subseteq V_G$, there corresponds a unique induced subgraph

$$G[A] = (A, E_G \cap E(A)) .$$

Also, to each subset $F \subseteq E_G$ of edges there corresponds a unique spanning subgraph of G,

$$G[F] = (V_G, F) .$$



G subgraph spanning induced

For a set $F \subseteq E_G$ of edges, we denote by $G - F$ the subgraph of G obtained by removing (only) the edges $e \in F$ from G. In particular, $G - e$ is obtained from G by removing $e \in E_G$. Clearly,

$$G - F = G[E_G - F] .$$

Similarly, we write $G + F$, if each $e \in F$ $(F \subseteq E(V_G))$ is added to G.

For a subset $A \subseteq V_G$ of vertices, we let $G - A$ be the subgraph of G induced by $V_G - A$, that is,

$$G - A = G[V_G - A] ,$$

and, *e.g.*, $G - v$ is obtained from G by removing the vertex v together with the edges that have v as their end.

Research problem: The famous open problem, **Ulam's problem** or the **Reconstruction Conjecture** (1929), states that *a graph is determined up to isomorphism by its vertex deleted subgraphs* $G{-}v$ $(v \in V_G)$: if there exists a bijection $\alpha\colon V_G \to V_H$ such that $G{-}v \cong H{-}\alpha(v)$ for all v, then $G \cong H$.

Many problems concerning (induced) subgraphs are algorithmically difficult. *E.g.*, the problem to determine for a given integer k and a graph G, if G has a clique K_m with $m \geq k$ as a subgraph, is NP-complete. To find a maximal clique (a subgraph K_m of maximum order) is unlikely to be even in NP.

Walks, paths and cycles

DEFINITION. Let $e_i = u_i u_{i+1} \in E_G$ be edges of G for $i \in [1,k]$. Here e_i and e_{i+1} are compatible in the sense that e_i is adjacent to e_{i+1} for all $i \in [1, k-1]$. The sequence

$$W = e_1 e_2 \dots e_k$$

is a **walk of length** k **from** u_1 **to** u_{k+1}. □

We write, more informally,

$$W : u_1 \longrightarrow u_2 \longrightarrow \dots \longrightarrow u_k \longrightarrow u_{k+1} \quad \text{or} \quad W : u_1 \xrightarrow{k} u_{k+1} \,.$$

Write $u \xrightarrow{\star} v$ to say that there is a walk of some length from u to v. Here we understand that $W\colon u \xrightarrow{\star} v$ *is always a specific walk*, $W = e_1 e_2 \dots e_k$, although we sometimes do not care to mention the edges e_i it uses. The length of a walk W is denoted by $|W|$.

DEFINITION. Let $W = e_1 e_2 \dots e_k$ $(e_i = u_i u_{i+1})$ be a walk.

W is **closed**, if $u_1 = u_{k+1}$.

W is a **path**, if $u_i \neq u_j$ for all $i \neq j$.

W is a **cycle**, if it is closed, and $u_i \neq u_j$ for $i \neq j$ except that $u_1 = u_{k+1}$.

W is a **trivial path**, if its length is 0. A trivial path has no edges.

For a walk $W : u = u_1 \longrightarrow \dots \longrightarrow u_{k+1} = v$, also

$$W^{-1}\colon v = u_{k+1} \longrightarrow \dots \longrightarrow u_1 = u$$

is a walk in G, called the **inverse walk** of W.

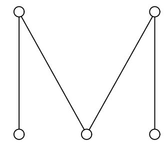
The **join** of two walks $W_1\colon u \xrightarrow{\star} v$ and $W_2\colon v \xrightarrow{\star} w$ is the walk $W_1 W_2\colon u \xrightarrow{\star} w$.

A vertex u is an **end** of a path P, if P starts or ends in u.

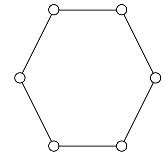
Paths P and Q are **disjoint**, if they have no vertices in common, and they are **independent**, if they can share only their ends. □

Clearly, the inverse walk P^{-1} of a path P is a path (the **inverse path** of P). The join of two paths need not be a path.

A (sub)graph, which is a path (cycle) of length $k-1$ (k, resp.) having k vertices is denoted by P_k (C_k, resp.). If k is even (odd), we say that the path or cycle is **even** (**odd**). Clearly, all paths of length k are isomorphic. The same holds for cycles of fixed length.



P_5



C_6

Lemma 1.5. *Each walk $W: u \xrightarrow{\star} v$ with $u \neq v$ contains a path $P: u \xrightarrow{\star} v$, that is, there is a path P that is obtained from W by removing edges and vertices.*

Proof. Let $W: u = u_1 \longrightarrow \dots \longrightarrow u_{k+1} = v$. Let $i < j$ be indices such that $u_i = u_j$. If no such i and j exist, then W, itself, is a path. Otherwise, in $W = W_1W_2W_3: u \xrightarrow{\star} u_i \xrightarrow{\star} u_j \xrightarrow{\star} v$ the portion $U_1 = W_1W_3: u \xrightarrow{\star} u_i = u_j \xrightarrow{\star} v$ is a shorter walk. By repeating this argument we obtain a sequence $U_1, U_2, \dots, U_m$ of walks $u \xrightarrow{\star} v$ with $|W| > |U_1| > \dots > |U_m|$. When the procedure stops, we have a path as required. (Notice that in the above it may very well be that W_1 or W_3 is a trivial walk.) □

DEFINITION. If there exists a walk (and hence a path) from u to v in G, let

$$d_G(u,v) = \min\{k \mid u \xrightarrow{k} v\}$$

be the **distance** between u and v. If there are no walks $u \xrightarrow{\star} v$, let $d_G(u,v) = \infty$ by convention. A graph G is **connected**, if $d_G(u,v) < \infty$ for all $u, v \in V_G$; otherwise, it is **disconnected**. The maximal connected subgraphs of G are its **components**. □

The maximality condition means that a subgraph $H \subseteq G$ is a component if and only if H is connected and for every vertex $v \notin V_H$, the subgraph $H+v$ is disconnected. Apparently, every component is an induced subgraph, and

$$N_G^*(v) = \{u \mid d_G(v,u) < \infty\}$$

is *the* component of G that contains $v \in V_G$. In particular, the components form a partition of G.

Denote

$$c(G) = \text{ the number of components of } G\,.$$

If $c(G) = 1$, then G is, of course, connected.

Shortest paths

DEFINITION. Let G^α be an edge weighted graph, that is, G^α is a graph G together with a weight function $\alpha\colon E_G \to \mathbb{R}$ on its edges. For a subgraph $H \subseteq G$, let

$$\alpha(H) = \sum_{e \in E_H} \alpha(e)$$

be the (total) **weight** of H. In particular, if $P = e_1 e_2 \dots e_k$ is a path, then its weight is $\alpha(P) = \sum_{i=1}^{k} \alpha(e_i)$. The **minimum weighted distance** between two vertices is

$$d_G^\alpha(u,v) = \min\{\alpha(P) \mid P\colon u \xrightarrow{\star} v\}\,. \qquad \square$$

In extremal problems we seek for an optimal subgraph $H \subseteq G$ satisfying specific conditions. In practice we encounter situations where G might represent

- a distribution or transportation network (say, for mail), where the weights on edges are *distances* or travel *expenses*;
- a system of channels in (tele)communication or computer architecture, where the weights denote *unreliability* of the connections.
- a model of chemical bonds with weights measuring the *attraction* of molecules.

In these examples we look for a subgraph with the smallest weight, and which connects two given vertices, or all vertices (if we want to travel around). On the other hand, if the graph represents a network of pipelines, the weights are volumes or capacities, and then one wants to find a subgraph with the maximum weight.

We consider the minimum problem. For this, let G be a graph with an integer weight function $\alpha\colon E_G \to \mathbb{N}$. In this case, call $\alpha(uv)$ the **length** of uv.

The shortest path problem: *Given a connected graph G with a weight function α, find $d_G^\alpha(u,v)$ for given $u, v \in V_G$.*

Assume that G is a connected graph. Dijkstra's algorithm solves the problem for every pair u, v, where u is a fixed starting point and $v \in V_G = V$. Let us make the convention that $\alpha(uv) = \infty$, if $uv \notin E_G$.

Dijkstra's algorithm:

(i) Set $u_0 = u$, $t(u_0) = 0$ and $t(v) = \infty$ for all $v \neq u_0$.

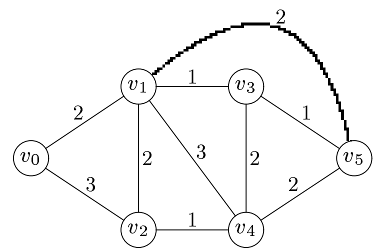
(ii) For $i \in [0, \nu_G - 1]$: for each $v \notin \{u_1, \dots, u_i\}$,

$$\text{replace} \quad t(v) \quad \text{by} \quad \min\{t(v), t(u_i) + \alpha(u_i v)\}\,.$$

Let $u_{i+1} \notin \{u_1, \dots, u_i\}$ be *any* vertex with the least value $t(u_{i+1})$.

(iii) Conclusion: $d_G^\alpha(u,v) = t(v)$.

As an example consider the following weighted graph G. Apply Dijkstra's algorithm to the vertex v_0.



- $u_0 = v_0$, $t(u_0) = 0$, others are ∞.
- $t(v_1) = \min\{\infty, 2\} = 2$, $t(v_2) = \min\{\infty, 3\} = 3$, others are ∞. Thus $u_1 = v_1$.
- $t(v_2) = \min\{3, t(u_1) + \alpha(u_1 v_2)\} = \min\{3, 4\} = 3$, $t(v_3) = 2+1 = 3$, $t(v_4) = 2+3 = 5$, $t(v_5) = 2+2 = 4$. Thus choose $u_2 = v_3$.
- $t(v_2) = \min\{3, \infty\} = 3$, $t(v_4) = \min\{5, 3+2\} = 5$, $t(v_5) = \min\{4, 3+1\} = 4$. Thus set $u_3 = v_2$.
- $t(v_4) = \min\{5, 3+1\} = 4$, $t(v_5) = \min\{4, \infty\} = 4$. Thus choose $u_4 = v_4$.
- $t(v_5) = \min\{4, 4+1\} = 4$. The algorithm stops.

We have obtained:

$$t(v_1) = 2,\ t(v_2) = 3,\ t(v_3) = 3,\ t(v_4) = 4,\ t(v_5) = 4\,.$$

These are the minimal weights from v_0 to each v_i.

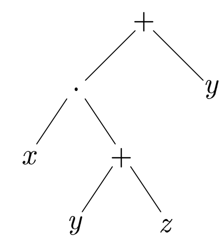
The correctness of the algorithm can be verified as follows. Suppose S is a proper subset of V. If $P\colon u_0 \xrightarrow{\star} u \xrightarrow{\star} v$ is a shortest path from u_0, which ends in a vertex of $V - S$ ($v \in V - S$), then clearly $u \in S$, and the path $u_0 \xrightarrow{\star} u$ must be a shortest path from u_0 to u. Therefore, the length of the path P equals the sum of the weights of $u_0 \xrightarrow{\star} u$ and $u \xrightarrow{\star} v$. This is exactly what Dijkstra's algorithm does.

2 Bipartite Graphs and Trees

In problems such as the shortest path problem we are looking for a set of solutions consisting of paths between given vertices. The solutions in these cases are usually subgraphs without cycles. Such (sub)graphs will be called trees, and they are used, *e.g.*, in search algorithms for databases. For concrete applications in this respect, see

T.H. CORMEN, C.E. LEISERSON AND R.L. RIVEST, "Introduction to Algorithms", MIT Press, 1993.

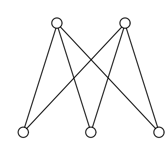
Certain structures with operations are representable as trees. These trees are sometimes called *construction trees*, *decomposition trees*, *factorization trees* or *grammatical trees*. Grammatical trees occur especially in linguistics, where syntactic structures of sentences are analyzed. On the right there is a tree of operations for the arithmetic formula $x \cdot (y + z) + y$.



Bipartite graphs

DEFINITION. A graph G is called **bipartite**, if V_G has a partition to two subsets X and Y such that each edge $uv \in E_G$ connects a vertex of X and a vertex of Y. In this case, (X, Y) is a **bipartition** of G, and G is (X, Y)-**bipartite**.

A bipartite graph G (as in the above) is a **complete** (m, k)-**bipartite graph**, if $|X| = m$, $|Y| = k$, and $uv \in E_G$ for all $u \in X$ and $v \in Y$.
All complete (m, k)-bipartite graphs are isomorphic. Let $K_{m,k}$ denote such a graph.
A subset $X \subseteq V_G$ is **stable**, if $G[X]$ is a discrete graph.



$K_{2,3}$ □

The following result is clear from the definitions.

Theorem 2.1. *A graph G is bipartite if and only if V_G has a partition to two stable subsets.*

Theorem 2.2. *A graph G is bipartite if and only if it has no odd cycles.*

Proof. (⇒) Let G be (X, Y)-bipartite. For a cycle $C\colon v_1 \longrightarrow \dots \longrightarrow v_{k+1} = v_1$ of length k, $v_1 \in X$ implies $v_2 \in Y$, $v_3 \in X$, ..., $v_{2i} \in Y$, $v_{2i+1} \in X$. Consequently, $k + 1 = 2m + 1$ is odd, and $k = |C|$ is even.

(⇐) Suppose that all cycles in G are even. First, we observe that it suffices to show the claim for connected graphs. Indeed, if G is disconnected, then each cycle of G is contained in one of the components, $G_1, \dots, G_p$, of G. If G_i is (X_i, Y_i)-bipartite, then $(X_1 \cup X_2 \cup \dots \cup X_p, Y_1 \cup Y_2 \cup \dots \cup Y_p)$ is a bipartition of G.

Assume thus that G is connected. Let $v \in V_G$ be a chosen vertex, and define

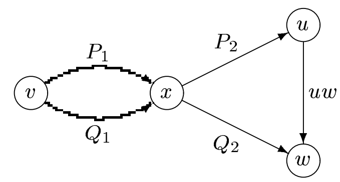
$$X = \{x \mid d_G(v,x) \text{ is even}\}, \qquad Y = \{y \mid d_G(v,y) \text{ is odd}\}.$$

Since G is connected, $V_G = X \cup Y$. Also, by the definition of distance, $X \cap Y = \emptyset$.

Let $u, w \in V_G$ be both in X or both in Y, and let $P: v \xrightarrow{\star} u$ and $Q: v \xrightarrow{\star} w$ be (among the) shortest paths from v to u and w.

Assume that x is the last common vertex of P and Q: $P = P_1P_2$, $Q = Q_1Q_2$, where $P_2: x \xrightarrow{\star} u$ and $Q_2: x \xrightarrow{\star} w$ are independent. Since P and Q are shortest paths, P_1 and Q_1 are shortest paths $v \xrightarrow{\star} x$.

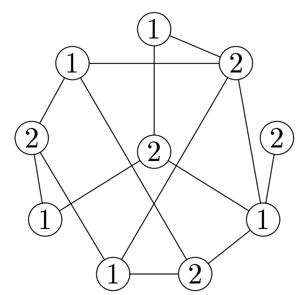
Consequently, $|P_1| = |Q_1|$, and so $|P_2|$ and $|Q_2|$ have the same parity. Therefore $Q_2^{-1}P_2: w \xrightarrow{\star} u$ is an even path. It follows that u and w are not adjacent in G, since otherwise $Q_2^{-1}P_2(uw)$ would be an odd cycle. Therefore $G[X]$ and $G[Y]$ are discrete induced subgraphs, and G is bipartite as claimed.



□

It is easy to check whether a graph is bipartite or not.

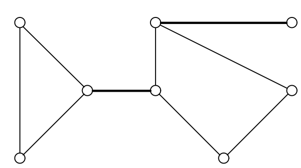
Indeed, this can be done using two 'opposite' colours, say 1 and 2. Start from any vertex v_1, and put colour 1 to it. Then colour the neighbours of v_1 by 2, and proceed by putting an opposite colour to all neighbours of an already coloured vertex. If the whole graph can be coloured, then G is (X,Y)-bipartite, where X consists of those vertices with colour 1, and Y of those vertices with colour 2; otherwise, at some point one of the vertices gets both colours, and in this case, G is not bipartite.



Bridges

DEFINITION. An edge $e \in E_G$ is a **bridge** of the graph G, if $c(G-e) > c(G)$. □

In particular, and most importantly, an edge e in a connected G is a bridge if and only if $G-e$ is disconnected. On the right the two horizontal lines are bridges. The rest are not.



Theorem 2.3. *An edge $e \in E_G$ is a bridge if and only if it is contained in no cycle of G.*

Proof. First of all, note that $e = uv$ is a bridge if and only if u and v belong to different components of $G{-}e$.

($\Rightarrow$) If there is a cycle in G containing e, then there is a cycle $C = eP \colon u \longrightarrow v \xrightarrow{\star} u$, where $P \colon v \xrightarrow{\star} u$ is a path in $G{-}e$, and so e is not a bridge.

($\Leftarrow$) Assume that $e = uv$ is not a bridge. Hence u and v are in the same component of $G{-}e$. If $P \colon v \xrightarrow{\star} u$ is a path in $G{-}e$, then $eP \colon u \longrightarrow v \xrightarrow{\star} u$ is a cycle in G that contains e. □

Lemma 2.4. *Let e be a bridge in a connected graph G.*

(i) $c(G{-}e) = 2$.

(ii) *Let H be a component of $G{-}e$. If $f \in E_H$ is a bridge of H, f is a bridge of G.*

Proof. For (i), let $e = uv$. Since e is a bridge, the ends u and v are not connected in $G{-}e$. Let $w \in V_G$. Since G is connected, there exists a path $P \colon w \xrightarrow{\star} v$ in G. This is a path of $G{-}e$, unless $P \colon w \xrightarrow{\star} u \to v$ contains $e = uv$, in which case the part $w \xrightarrow{\star} u$ is a path in $G{-}e$.

For (ii), if $f \in E_H$ belongs to a cycle C of G, then C does not contain e (since e is in no cycle), and therefore C is inside H, and f is not a bridge of H. □

Trees

DEFINITION. A graph is called **acyclic**, if it has no cycles. A **tree** is a connected acyclic graph. □

By Theorem 2.3 and the definition of a tree, we have

Corollary 2.5. *A connected graph is a tree if and only if all its edges are bridges.*

The following was proved by CAYLEY in 1889.

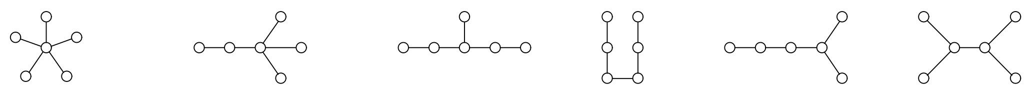
Theorem 2.6. *Let $|V| = n$. There are n^{n-2} trees on V.*

On the other hand, there are only a few trees *up to isomorphism*:

n	1	2	3	4	5	6	7	8
trees	1	1	1	2	3	6	11	23

n	9	10	11	12	13	14	15	16
trees	47	106	235	551	1301	3159	7741	19 320

The nonisomorphic trees of order 6 are:



Theorem 2.7. *The following are equivalent for a graph T.*

(i) *T is a tree.*
(ii) *Any two vertices are connected in T by a unique path.*
(iii) *T is acyclic and $\varepsilon_T = \nu_T - 1$.*

Proof. Let $\nu_T = n$. If $n = 1$, then the claim is trivial. Suppose thus that $n \geq 2$.

(i)⇒(ii) Let T be a tree. Assume the claim does not hold, and choose $u, v \in V_T$ to be of minimum distance $d_T(u, v)$ such that there are two different paths $P, Q\colon u \xrightarrow{\star} v$. We may assume that $|P| = d_T(u, v)$. If P and Q are independent, then PQ^{-1} is a cycle, and this is a contradiction. Otherwise, let $x \notin \{u, v\}$ be a common vertex in these paths, $P = P_1P_2$ and $Q = Q_1Q_2\colon u \xrightarrow{\star} x \xrightarrow{\star} v$. Now there are two different paths $P_1, Q_1\colon u \xrightarrow{\star} x$ or $P_2, Q_2\colon x \xrightarrow{\star} v$, where $|P_1| < |P|$ and $|P_2| < |P|$. This contradicts the minimality assumption.

(ii)⇒(iii) We prove the claim by induction on n. Clearly, the claim holds for $n = 2$, and suppose it holds for graphs of order less than n. Let T be any graph of order n satisfying (ii). In particular, T is connected.

Let $P\colon u \xrightarrow{\star} v$ be a *maximal path* in T, that is, there are no edges e, for which Pe or eP is a path. Such paths exist, because ν_T is finite. It follows that $d_T(v) = 1$, since, by maximality, if $vw \in E_T$, then w belongs to P; otherwise $P(vw)$ would be a longer path. In this case, $P\colon u \xrightarrow{\star} w \longrightarrow v$, where vw is the unique edge having an end v. The subgraph $T-v$ is connected, and therefore it satisfies the condition (ii). By induction hypothesis, $\varepsilon_{T-v} = n - 2$, and so $\varepsilon_T = \varepsilon_{T-v} + 1 = n - 1$, and the claim follows.

(iii)⇒(i) Assume (iii) holds for T. We need to show that T is connected. Indeed, let the components of T be $T_i = (V_i, E_i)$, for $i \in [1, k]$. Since T is acyclic, so are the connected graphs T_i, and hence they are trees, for which we have proved that $|E_i| = |V_i| - 1$. Now, $\nu_T = \sum_{i=1}^{k} |V_i|$, and $\varepsilon_T = \sum_{i=1}^{k} |E_i|$. Therefore,

$$n - 1 = \varepsilon_T = \sum_{i=1}^{k} (|V_i| - 1) = \sum_{i=1}^{k} |V_i| - k = n - k\,,$$

which gives that $k = 1$, that is, T is connected. □

Spanning trees

Theorem 2.8. *Each connected graph has a spanning tree, that is, a spanning graph that is a tree.*

Proof. Let H be a minimal connected spanning subgraph of G, that is, a connected spanning subgraph of G such that $H{-}e$ is disconnected for all $e \in E_H$. Such a subgraph is obtained from G by removing nonbridges:

- To start with, let $H_0 = G$.
- For $i \geq 0$, let $H_{i+1} = H_i{-}e_i$, where e_i is a not a bridge of H_i. Since e_i is not a bridge, H_{i+1} is a connected spanning subgraph of H_i and thus of G.
- $H = H_k$, when only bridges are left.

By Corollary 2.5, H is a tree. □

Corollary 2.9. *For each connected graph G, $\varepsilon_G \geq \nu_G - 1$. Moreover, a connected graph G is a tree if and only if $\varepsilon_G = \nu_G - 1$.*

Proof. Let T be a spanning tree of G. Then $\varepsilon_G \geq \varepsilon_T = \nu_T - 1 = \nu_G - 1$. The second claim is also clear. □

Corollary 2.10. *Each nontrivial tree T has at least two leaves.*

Proof. Let ℓ be the number of leaves of T. By Corollary 2.9 and the handshaking lemma,

$$\begin{aligned} 2 \cdot \nu_T - 2 = 2 \cdot \varepsilon_T &= \sum_{v \in V_T} d_T(v) = \sum_{d_T(v) > 1} d_T(v) + \ell \\ &\geq 2 \cdot (\nu_T - \ell) + \ell = 2 \cdot \nu_T - \ell\,, \end{aligned}$$

from which it follows that $\ell \geq 2$, as required. □

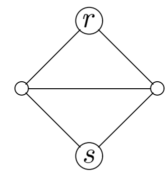
Example 2.11. In **Shannon's game** a positive player P and a negative player N play on a graph G with two special vertices: a **source** s and a **sink** r. P and N alternate turns so that P designates an edge by $+$, and N by $-$. Each edge can be designated at most once. It is P's purpose to designate a path $s \xrightarrow{\star} r$ (that is, to designate all edges in one such path), and N tries to block all paths $s \xrightarrow{\star} r$ (that is, to designate at least one edge in each such path).

We say that a game (G, s, r) is

- **positive**, if P has a winning strategy no matter who begins the game,
- **negative**, if N has a winning strategy no matter who begins the game,
- **neutral**, if the winner depends on who begins the game.

The game on the right is neutral.

Whether a game is positive or not was solved by LEHMAN:



A Shannon's game (G, s, r) is positive if and only if G has a subgraph H containing s and r such that H has two spanning trees with no edges in common.

There remains the problem to characterize those Shannon's games (G, s, r) that are neutral (negative, respectively). □

The connector problem

To build a network connecting n nodes (towns, computers, chips in a computer) it is desirable to decrease the cost of construction of the links to the minimum. This is the **connector problem**. In graph theoretical terms we wish to find an **optimal spanning subgraph** of a weighted graph. Such an optimal subgraph is clearly a spanning tree, for, otherwise a deletion of any nonbridge will reduce the total weight of the subgraph.

Let then G^α be a graph G together with a weight function $\alpha\colon E_G \to \mathbb{R}^+$ (positive reals) on the edges. Kruskal's algorithm (also known as the **greedy algorithm**) provides a solution to the connector problem.

Kruskal's algorithm: For a connected and weighted graph G^α of order n:

(i) Let e_1 be an edge of smallest weight, and set $E_1 = \{e_1\}$.

(ii) For each $i = 2, 3, \dots, n-1$ in this order, choose an edge $e_i \notin E_{i-1}$ of smallest possible weight such that e_i does not produce a cycle when added to $G[E_{i-1}]$, and let $E_i = E_{i-1} \cup \{e_i\}$.

The final outcome is $T = (V_G, E_{n-1})$.

By the construction, $T = (V_G, E_{n-1})$ is a spanning tree of G, because it contains no cycles, it is connected and has $n-1$ edges. We now show that T has the minimum total weight among the spanning trees of G.

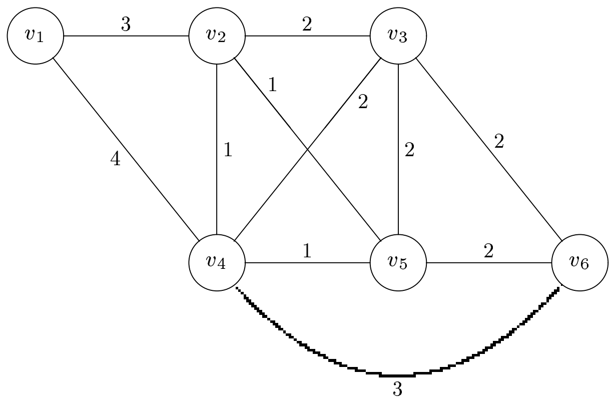
Suppose T_1 is any spanning tree of G. Let e_k be the first edge produced by the algorithm that is not in T_1. If we add e_k to T_1, then a cycle C containing e_k is created. Also, C must contain an edge e that is not in T. When we replace e by e_k in T_1, we still have a spanning tree, say T_2. However, by the construction, $\alpha(e_k) \le \alpha(e)$, and therefore $\alpha(T_2) \le \alpha(T_1)$. Note that T_2 has more edges in common with T than T_1.

Repeating the above procedure, we can transform T_1 to T by replacing edges, one by one, such that the total weight does not increase. We deduce that $\alpha(T) \le \alpha(T_1)$.

The outcome of Kruskal's algorithm need not be unique. Indeed, there may exist several optimal spanning trees (with the same weight, of course) for a graph.

When applied to the weighted graph on the right, the algorithm produces the sequence: $e_1 = v_2v_4$, $e_2 = v_4v_5$, $e_3 = v_3v_6$, $e_4 = v_2v_3$ and $e_5 = v_1v_2$. The total weight of the spanning tree is thus 9.

Also, the selection $e_1 = v_2v_5$, $e_2 = v_4v_5$, $e_3 = v_5v_6$, $e_4 = v_3v_6$, $e_5 = v_1v_2$ gives another optimal solution (of weight 9).

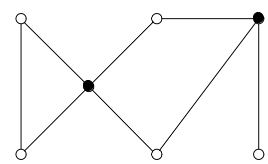


3 Connectivity

Spanning trees are often optimal solutions to problems, where cost is the criterion. We may also wish to construct graphs that are as simple as possible, but where two vertices are always connected by at least two independent paths. Such problems occur especially in the theory of reliable networks, where one has to make sure that a break-down of one connection does not affect the functionality of the network. Similarly, in a reliable network we require that a break-down of a node (computer) should not result in the inactivity of the whole network.

Separating sets

DEFINITION. A vertex $v \in V_G$ is a **cut vertex**, if $c(G-v) > c(G)$. A subset $A \subseteq V_G$ is a **separating set**, if $G-A$ is disconnected. □



If G is connected, then v is a cut vertex if and only if $G-v$ is disconnected, that is, $\{v\}$ is a separating set.

Lemma 3.1. *If a connected graph G has no separating sets, then it is a clique.*

Proof. If $\nu_G \leq 2$, then the claim is clear. For $\nu_G \geq 3$, assume that G is not a clique, and let $uv \notin E_G$. Now $V_G - \{u, v\}$ is a separating set. The claim follows from this. □

DEFINITION. The (**vertex**) **connectivity number** $\kappa(G)$ of G is defined as

$$\kappa(G) = \min\{k \mid k = |A|,\ G-A \text{ disconnected or trivial}, A \subseteq V_G\}\,.$$

A graph G is k-**connected**, if $\kappa(G) \geq k$. □

In other words,

- $\kappa(G) = 0$, if G is disconnected,
- $\kappa(G) = \nu_G - 1$, if G is a clique, and
- otherwise $\kappa(G)$ equals the minimum size of a separating set of G.

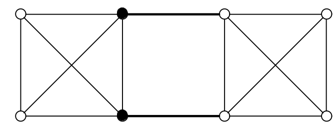
Clearly, if G is connected, then it is 1-connected.

DEFINITION. An **edge cut** F of G consists of edges so that $G-F$ is disconnected. Let

$$\kappa'(G) = \min\{k \mid k = |F|,\ G-F \text{ disconnected}, F \subseteq E_G\}\,.$$

For trivial graphs, let $\kappa'(G) = 0$. A graph G is k-**edge connected**, if $\kappa'(G) \geq k$. A minimal edge cut $F \subseteq E_G$ is a **bond** ($F - e$ is not an edge cut for any $e \in F$). □

Again, if G is disconnected, then $\kappa'(G) = 0$.
On the right, $\kappa(G) = 2$ and $\kappa'(G) = 2$. Notice that the minimum degree is $\delta(G) = 3$.



Lemma 3.2. *Let G be connected. If $e = uv$ is a bridge, then either $G = K_2$ or one of u or v is a cut vertex.*

Proof. Assume that $G \neq K_2$ and thus that $\nu_G \geq 3$, since G is connected. Let $G_u = N^*_{G-e}(u)$ and $G_v = N^*_{G-e}(v)$ be the components of $G-e$ containing u and v. Now, either $|V_{G_u}| \geq 2$ (and u is a cut vertex) or $|V_{G_v}| \geq 2$ (and v is a cut vertex). □

Lemma 3.3. *Let F be a bond of a connected graph G. Then $G-F$ has exactly two components.*

Proof. Exercise. □

The following result is due to Whitney (1932).

Theorem 3.4. *For any graph G,*

$$\kappa(G) \leq \kappa'(G) \leq \delta(G)\,.$$

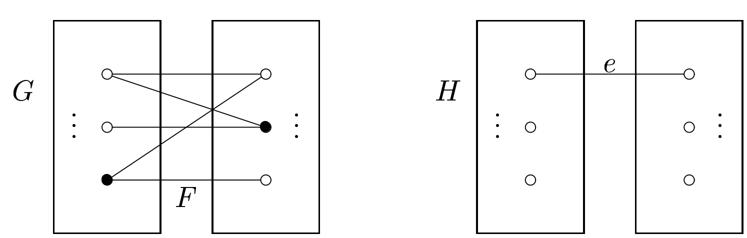
Proof. Assume G is nontrivial. Clearly, $\kappa'(G) \leq \delta(G)$, since if we remove all edges with an end v, we disconnect G.

If $\kappa'(G) = 0$, then G is disconnected, and in this case also $\kappa(G) = 0$. If $\kappa'(G) = 1$, then G is connected and contains a bridge. By Lemma 3.2, either $G = K_2$ or G has a cut vertex. In both of these cases, also $\kappa(G) = 1$.

Assume then that $\kappa'(G) \geq 2$. Let F be an edge cut of G with $|F| = \kappa'(G)$, and let $e = uv \in F$. Then F is a bond, and $G-F$ has two components. Consider the connected subgraph

$$H = G-(F-\{e\}) = (G-F) + e\,,$$

where e is a bridge. Now for each $f \in F-\{e\}$ select an end in G different from u and v (since $f \neq e$, either end of f is different from u or v). These choices are not assumed to be different, that is, you may choose the same end for two adjacent edges. Let S be the collection of these choices. Thus $|S| \leq |F| - 1 = \kappa'(G) - 1$, and $G-S$ does not contain edges from $F-\{e\}$.



If $G-S$ is disconnected, then S is a separating set and so $\kappa(G) \le |S| \le \kappa'(G) - 1$ and we are done. On the other hand, if $G-S$ is connected, then either $G-S = K_2$ $(= e)$, or either u or v (or both) is a cut vertex of $G-S$ (since $H-S = G-S$, and therefore $G-S$ is an induced subgraph of H). In both of these cases, there is a vertex of $G-S$, whose removal results in a trivial or a disconnected graph. In conclusion, $\kappa(G) \le |S| + 1 \le \kappa'(G)$, and the claim follows. □

Menger's theorem

Definition. Let $u, v \in V_G$ for a graph G. A subset $S \subseteq V_G - \{u, v\}$ is said to **separate** u and v, if there are no paths $u \xrightarrow{\star} v$ in $G-S$. □

Theorem 3.5. *Let $u, v \in V_G$ be nonadjacent vertices of a connected graph G. Then the minimum number of vertices separating u and v is equal to the maximum number of independent paths from u to v.*

Proof. If a subset $S \subseteq V_G$ separates u and v, then every path $u \xrightarrow{\star} v$ of G visits at least one vertex of S. Hence $|S|$ is at least the number of independent paths from u to v.

In the other direction, we use induction on $m = \nu_G + \varepsilon_G$ to show that if

$$S = \{w_1, w_2, \dots, w_k\}$$

is a *minimum set* (that is, a subset of the smallest size) that separates u and v, then G has at least (and thus exactly) k independent paths $u \xrightarrow{\star} v$.

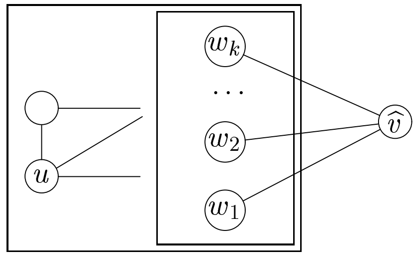
The case for $k = 1$ is clear, and this takes care of the small values of m, required for the induction.

(1) Assume first that u and v have a common neighbour $w \in N_G(u) \cap N_G(v)$. Then necessarily $w \in S$. In the smaller graph $G-w$ the set $S - \{w\}$ is a minimum set that separates u and v, and so the induction hypothesis yields that there are $k-1$ independent paths $u \xrightarrow{\star} v$ in $G-w$. Together with the path $u \longrightarrow w \longrightarrow v$, there are k independent paths $u \xrightarrow{\star} v$ in G as required.

(2) Assume then that $N_G(u) \cap N_G(v) = \emptyset$, and denote by $H_u = N^*_{G-S}(u)$ and $H_v = N^*_{G-S}(v)$ the components of $G-S$ for u and v.

(2.1) Suppose next that $S \nsubseteq N_G(u)$ and $S \nsubseteq N_G(v)$.

Let $\widehat{v}$ be a new vertex, and define G_u to be the graph on $H_u \cup S \cup \{\widehat{v}\}$ having the edges of $G[H_u \cup S]$ together with $\widehat{v}w_i$ for all $i \in [1, k]$. The graph G_u is connected and it is smaller than G. Indeed, in order for S to be a minimum separating set, all $w_i \in S$ have to be adjacent to some vertex in H_v. This shows that $\varepsilon_{G_u} \le \varepsilon_G$, and, moreover, the assumption (2.1) rules out the case $H_v = \{v\}$, and therefore $|H_v| \ge 2$ and so $\nu_{G_u} < \nu_G$ in the present case.



If S' is any subset that separates u and $\widehat{v}$ in G_u, then S' will separate u from all $w_i \in S - S'$ in G. This means that S' separates u and v in G. Since k is the size of a minimum separating set, $|S'| \geq k$. We noted that G_u is smaller than G, and thus by the induction hypothesis, there are k independent paths $u \xrightarrow{\star} \widehat{v}$ in G_u. This is possible only if there exist k paths $u \xrightarrow{\star} w_i$, one for each $i \in [1, k]$, that have only the end u in common.

By the present assumption, also u is nonadjacent to some vertex of S. A symmetric argument applies to the graph G_v (with a new vertex $\widehat{u}$) , which is defined similarly to G_u. This yields that there are k paths $w_i \xrightarrow{\star} v$ that have only the end v in common. When we combine these with the above paths $u \xrightarrow{\star} w_i$, we obtain k independent paths $u \xrightarrow{\star} w_i \xrightarrow{\star} v$ in G.

(2.2) There remains the case, where for *all* separating sets S of k elements, either $S \subseteq N_G(u)$ or $S \subseteq N_G(v)$. (Note that then, by (2), $S \cap N_G(v) = \emptyset$ or $S \cap N_G(u) = \emptyset$.)

Let $P = efQ$ be a shortest path $u \xrightarrow{\star} v$ in G, where $e = ux$, $f = xy$, and $Q\colon y \xrightarrow{\star} v$. Notice that, by the assumption (2), $|P| \geq 3$, and so $y \neq v$. In the smaller graph $G{-}f$, let S' be a minimum set that separates u and v.

If $|S'| \geq k$, then, by the induction hypothesis, there are k independent paths $u \xrightarrow{\star} v$ in $G{-}f$. But these are paths of G, and the claim is clear in this case.

If, on the other hand, $|S'| < k$, then u and v are still connected in $G{-}S'$. Indeed, every path $u \xrightarrow{\star} v$ in $G{-}S'$ necessarily travels along the edge $f = xy$, and so $x, y \notin S'$. Let

$$S_x = S' \cup \{x\} \qquad \text{and} \qquad S_y = S' \cup \{y\}\,.$$

These sets separate u and v in G (by the above fact), and they have size k. By our current assumption, the vertices of S_y are adjacent to v, since the path P is shortest and so $uy \notin E_G$ (meaning that u is not adjacent to all of S_y). The assumption (2) yields that u is adjacent to all of S_x, since $ux \in E_G$. But now both u and v are adjacent to the vertices of S', which contradicts the assumption (2). □

Theorem 3.6. *A graph G is k-connected if and only if every two vertices are connected by at least k independent paths.*

Proof. If any two vertices are connected by k independent paths, then it is clear that $\kappa(G) \geq k$.

In converse, suppose that $\kappa(G) = k$, but that G has vertices u and v connected by at most $k-1$ independent paths. By Theorem 3.5, it must be that $e = uv \in E_G$. Consider the graph $G{-}e$. Now u and v are connected by at most $k-2$ independent paths in $G{-}e$, and by Theorem 3.5, u and v can be separated by a set S with $|S| = k-2$. Since $\nu_G > k$ (because $\kappa(G) = k$), there exists a $w \in V_G$ that is not in $S \cup \{u, v\}$. The vertex w is separated by S from u or from v; otherwise there would be a path $u \xrightarrow{\star} v$ in $(G{-}e){-}S$. Say, this vertex is u. The set $S \cup \{v\}$ has $k-1$ elements, and it separates u from w in G, which contradicts the assumption that $\kappa(G) = k$. This proves the claim. □

We state without a proof the corresponding separation property for edge connectivity.

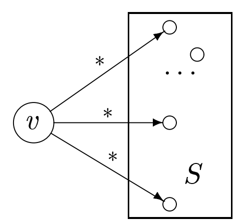
DEFINITION. Let G be a graph. A uv-**disconnecting set** is a set $F \subseteq E_G$ such that every path $u \xrightarrow{\star} v$ contains an edge from F. □

Theorem 3.7. *Let $u, v \in V_G$ with $u \neq v$ in a graph G. Then the maximum number of edge-disjoint paths $u \xrightarrow{\star} v$ equals the minimum number k of edges in a uv-disconnecting set.*

Corollary 3.8. *A graph G is k-edge connected if and only if every two vertices are connected by at least k edge disjoint paths.*

Dirac's fans

DEFINITION. Let $v \in V_G$ and $S \subseteq V_G$ such that $v \notin S$ in a graph G. A set of paths from v to a vertex in S is called a (v, S)-**fan**, if they have only v in common. □



DIRAC (1960) proved

Theorem 3.9. *A graph G is k-connected if and only if $\nu_G > k$ and for every $v \in V_G$ and $S \subseteq V_G$ with $|S| \geq k$ and $v \notin S$, there exists a (v, S)-fan of k paths.*

Proof. Exercise. □

Theorem 3.10. *Let G be a k-connected graph for $k \geq 2$. Then for any k vertices, there exists a cycle of G containing them.*

Proof. First of all, since $\kappa(G) \geq 2$, G has no cut vertices, and thus no bridges. It follows that every edge, and thus every vertex of G belongs to a cycle.

Let $S \subseteq V_G$ be such that $|S| = k$, and let C be a cycle of G that contains the maximum number of vertices of S. Let the vertices of $S \cap V_C$ be $v_1, \dots, v_r$ listed in order around C so that each pair (v_i, v_{i+1}) (with indices modulo r) defines a path along C (except in the special case where $r = 1$). Such a path is referred to as a *segment* of C. If C contains all vertices of S, then we are done; otherwise, suppose $v \in S$ is not on C.

It follows from Theorem 3.9 that there is a (v, V_C)-fan of at least $\min\{k, |V_C|\}$ paths. Therefore there are two paths $P\colon v \xrightarrow{\star} u$ and $Q\colon v \xrightarrow{\star} w$ in such a fan that end in the same segment (v_i, v_{i+1}) of C. Then the path $W\colon u \xrightarrow{\star} w$ (or $w \xrightarrow{\star} u$) along C contains all vertices of $S \cap V_C$. But now PWQ^{-1} is a cycle of G that contains v and all v_i for $i \in [1, r]$. This contradicts the choice of C, and proves the claim. $\square$

4 Eulerian and Hamiltonian Graphs

In the connector problem we reduced the cost of a spanning graph to its minimum. There are different problems, where the cost is measured by an active user of the graph. For instance, in the **travelling salesman problem** a person is supposed to visit each town in his district, and this he should do in such a way that saves time and money. Obviously, he should plan the travel so as to visit each town once, and so that the overall flight time is as short as possible. In graph theoretical terms, we look for a minimum weighted Hamilton cycle of a graph, the vertices of which are the towns and the weights on the edges are the flight times. Unlike for the shortest path and the connector problems no reliable and efficient algorithm is known for the travelling salesman problem. Indeed, it is widely believed that no practical algorithm exists for this problem.

Eulerian graphs

DEFINITION. A walk $W = e_1 e_2 \dots e_n$ is a **trail**, if $e_i \neq e_j$ for all $i \neq j$. A connected graph G is **Eulerian**, if it has a closed trail containing every edge of G. Such a trail is called an **Euler tour**. □

Notice that if $W = e_1 e_2 \dots e_n$ is an Euler tour (and so $E_G = \{e_1, e_2, \dots, e_n\}$), also $e_i e_{i+1} \dots e_n e_1 \dots e_{i-1}$ is an Euler tour for all $i \in [1, n]$. A complete proof of the following **Euler's Theorem** (1736) was first given by HIERHOLZER (1873).

Theorem 4.1. *A connected graph G is Eulerian if and only if every vertex has an even degree.*

Proof. ($\Rightarrow$) Suppose $W : u \xrightarrow{\star} u$ is an Euler tour. Let v ($\neq u$) be a vertex that occurs k times in W. Every time an edge arrives at v, another edge departs from v, and therefore $d_G(v) = 2k$. Also, $d_G(u)$ is even, since W starts and ends at u.

($\Leftarrow$) Assume G is a nontrivial connected graph such that $d_G(v)$ is even for all $v \in V_G$. Let

$$W = e_1 e_2 \dots e_n \colon v_0 \xrightarrow{\star} v_n \quad \text{with} \quad e_i = v_{i-1} v_i$$

be a longest trail in G. It follows that all $e = v_n w \in E_G$ are among the edges of W, for, otherwise, W could be prolonged to We. In particular, $v_0 = v_n$, that is, W is a closed trail. (Indeed, if it were $v_n \neq v_0$ and v_n occurs k times in W, then $d_G(v_n) = 2(k-1)+1$ and that would be odd.)

If W is not an Euler tour, then, since G is connected, there exists an edge $f = v_i u \in E_G$ for some i, which is not in W. However, now

$$e_{i+1} \dots e_n e_1 \dots e_i f$$

is a trail in G, and it is longer than W. This contradiction to the choice of W proves the claim. □

DEFINITION. An **Euler trail** of a graph G is a trail that visits every edge once. □

An Euler trail need not be closed.

Theorem 4.2. *A connected graph has an Euler trail if and only if it has at most two vertices of odd degree.*

Proof. If G has an Euler trail $u \xrightarrow{\star} v$, then, as in the proof of Theorem 4.1, each vertex $w \notin \{u, v\}$ has an even degree.

Assume then that G is connected and has at most two vertices of odd degree. If G has no vertices of odd degree then, by Theorem 4.1, G has an Euler trail. Otherwise, by the handshaking lemma, every graph has an even number of vertices with odd degree, and therefore G has exactly two such vertices, say u and v. Now, as in the previous proof, a longest trail of G is necessarily $u \xrightarrow{\star} v$ with $e = uv \notin E_G$. In $G + e$ every vertex has an even degree, and hence there is an Euler tour in $G + e$. When e is removed from this tour, we obtain an Euler trail of G. □

Example 4.3. The following problem is due to GUAN MEIGU (1962). Consider a village, where a postman wishes to plan his route to save the legs, but still every street has to be walked through. This problem is akin to Euler's problem and to the shortest path problem.

Let G be a graph with a weight function $\alpha\colon E_G \to \mathbb{R}^+$. The **chinese postman problem** is to find a minimum weighted tour in G (starting from a given vertex, the post office).

If G is *Eulerian*, then any Euler tour will do as a solution, because such a tour traverses each edge exactly once and this is the best one can do. In this case the weight of the optimal tour is the total weight of the graph G, and there is a good algorithm for finding such a tour:

Fleury's algorithm:

- Let $v_0 \in V_G$ be a chosen vertex, and let W_0 be the trivial path on v_0.
- Repeat the following procedure for $i = 1, 2, \dots$ as long as possible: suppose a trail $W_i = e_1 e_2 \dots e_i$ has been constructed, where $e_j = v_{j-1} v_j$.
  Choose an edge e_{i+1} ($\neq e_j$ for $j \in [1, i]$) so that

(i) e_{i+1} has an end v_i, and
(ii) e_{i+1} is not a bridge of $G_i = G - \{e_1, \dots, e_i\}$, unless there is no alternative.

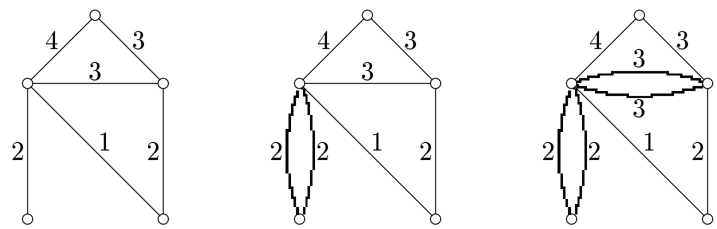
Notice that, as is natural, the weights $\alpha(e)$ play no role in the Eulerian case.

As an exercise we state:

If G is Eulerian, then any trail of G constructed by Fleury's algorithm is an Euler tour of G.

If G is *not Eulerian*, the poor postman has to walk at least one street twice. This happens, *e.g.*, if one of the streets is a dead end, and in general if there is a street corner of an odd number of streets. We can attack this case by reducing it to the

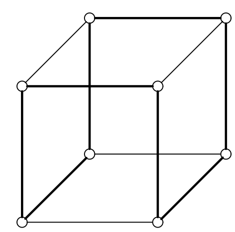
Eulerian case as follows. An edge $e = uv$ will be **duplicated**, if it is added to G parallel to an existing edge $e' = uv$ with the same weight, $\alpha(e') = \alpha(e)$.



Above we have duplicated two edges. The rightmost (multi)graph is Eulerian.

There is a good algorithm by Edmonds and Johnson (1973) for the construction of an optimal Eulerian supergraph by duplications. Unfortunately, this algorithm is somewhat complicated, and we shall skip it. □

Hamiltonian graphs



Definition. A path P of a graph G is a **Hamilton path**, if P visits every vertex of G once. Similarly, a cycle C is a **Hamilton cycle**, if it visits each vertex once. A graph is **Hamiltonian**, if it has a Hamilton cycle. □

Note that if $C: u_1 \to u_2 \to \cdots \to u_n$ is a Hamilton cycle, then so is $u_i \to \dots u_n \to u_1 \to \dots u_{i-1}$ for each $i \in [1, n]$, and thus we can choose where to start the cycle.

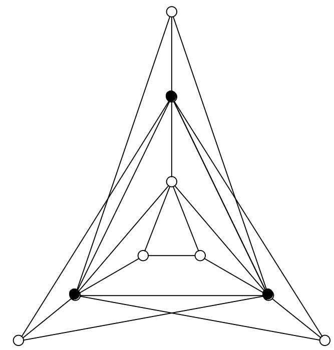
Example 4.4. It is obvious that each clique K_n is Hamiltonian whenever $n \geq 3$. □

Unlike for Eulerian graphs (Theorem 4.1) no good characterization is known for Hamiltonian graphs. Indeed, the problem to determine if G is Hamiltonian is NP-complete. There are, however, some interesting general conditions.

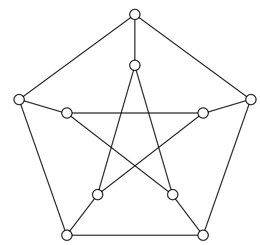
Lemma 4.5. *If G is Hamiltonian, then for every nonempty subset $S \subseteq V_G$,*

$$c(G{-}S) \leq |S| .$$

Proof. Let $\emptyset \neq S \subseteq V_G$, $u \in S$, and let $C\colon u \xrightarrow{\star} u$ be a Hamilton cycle of G. Assume $G{-}S$ has k components, G_i, $i \in [1, k]$. The case $k = 1$ is trivial, and hence we may suppose that $k > 1$. Let u_i be the last vertex of C that belongs to G_i, and let v_i be the vertex that follows u_i in C. Now $v_i \in S$ for each i by the choice of u_i, and $v_j \neq v_t$ for all $j \neq t$, because C is a cycle and $u_i v_i \in E_G$ for all i. This means that $|S| \geq k$ as required. □



This result can be used to show that some graphs are *not* Hamiltonian. As an example, consider the graph on the right. Choose S to be the subset of black vertices. Then $G-S$ has four components, and by Lemma 4.5, G is not Hamiltonian.



Consider next the **Petersen graph**, which appears in many places in graph theory (as a counterexample for various conditions). This graph is not Hamiltonian, but it satisfies the conclusion of Lemma 4.5. Therefore the conclusion of Lemma 4.5 is not sufficient to ensure that a graph is Hamiltonian.

The following theorems are due to ORE (1962) and they generalize an earlier result by DIRAC (1952), see Corollary 4.10.

Theorem 4.6. *Let G be a graph of order $\nu_G \geq 3$, and let $u, v \in V_G$ be such that*

$$d_G(u) + d_G(v) \geq \nu_G\,.$$

Then G is Hamiltonian if and only if $G + uv$ is Hamiltonian.

Proof. Denote $n = \nu_G$. Let $u, v \in V_G$ be such that $d_G(u) + d_G(v) \geq n$. If $uv \in E_G$, then there is nothing to prove. Assume thus that $uv \notin E_G$.

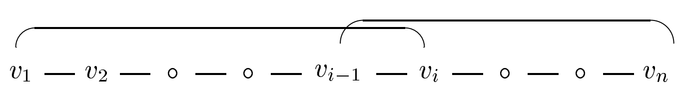
($\Rightarrow$) This is trivial since if G has a Hamilton cycle C, then C is also a Hamilton cycle of $G + uv$.

($\Leftarrow$) Denote $e = uv$ and suppose that $G + e$ has a Hamilton cycle C. If C does not use the edge e, then it is a Hamilton cycle of G. Suppose thus that e is on C. We may then assume that $C \colon u \xrightarrow{\star} v \longrightarrow u$. Now $u = v_1 \longrightarrow v_2 \longrightarrow \ldots \longrightarrow v_n = v$ is a Hamilton *path* of G.

There exists an i with $1 < i < n$ such that

$$uv_i \in E_G \;\text{ and }\; v_{i-1}v \in E_G\,.$$

For, otherwise $d_G(v) < n - d_G(u)$ would contradict the assumption.



But now $u = v_1 \longrightarrow \ldots \longrightarrow v_{i-1} \longrightarrow v_n \longrightarrow v_{n-1} \longrightarrow \ldots \longrightarrow v_{i+1} \longrightarrow v_i \longrightarrow v_1 = u$ is a Hamilton cycle in G. □

Closure

Define the **closure** of a graph G inductively as follows. Let $\nu_G = n$, and define

$$G_0 = G \text{ and } G_{i+1} = G_i + uv\,,$$

where u and v are any vertices such that $uv \notin E_{G_i}$ and $d_{G_i}(u) + d_{G_i}(v) \geq n$. This procedure stops when no new edges can be added to G_k for some k: for all $u, v \in V_G$ either $uv \in E_{G_k}$ or $d_{G_k}(u) + d_{G_k}(v) < n$. We shall denote by $cl(G)$ ($= G_k$) the result of this procedure.

In each step of the construction of $cl(G)$ there are usually many choices which edge uv is to be added to the graph, and therefore the above procedure is not deterministic. However, the *final result* $cl(G)$ is independent of the choices:

Lemma 4.7. *The closure $cl(G)$ is uniquely defined for all graphs G of order $\nu_G \geq 3$.*

Proof. Exercise □

Theorem 4.8. *Let G be a graph of order $\nu_G \geq 3$.*

(i) *G is Hamiltonian if and only if its closure $cl(G)$ is Hamiltonian.*
(ii) *If $cl(G)$ is a clique, then G is Hamiltonian.*

Proof. First, $G \subseteq cl(G)$ and G spans $cl(G)$, and thus if G is Hamiltonian, so is $cl(G)$.

In the other direction, let $G = G_0, G_1, \ldots, G_k = cl(G)$ be a construction sequence of the closure of G. If $cl(G)$ is Hamiltonian, then so are $G_{k-1}, \ldots, G_1$ and G_0 by Theorem 4.6.

The Claim (ii) follows from (i), since each clique is Hamiltonian. □

Theorem 4.9. *Let G be a graph of order $\nu_G \geq 3$. Suppose that for all nonadjacent vertices u and v, $d_G(u) + d_G(v) \geq \nu_G$. Then G is Hamiltonian.*

Proof. Since $d_G(u) + d_G(v) \geq \nu_G$ for all nonadjacent vertices, we have $cl(G) = K_n$ for $n = \nu_G$, and thus G is Hamiltonian. □

Corollary 4.10. *Let G be a graph of order $\nu_G \geq 3$, and assume that $\delta(G) \geq \frac{1}{2}\nu_G$. Then G is Hamiltonian.*

Proof. This is immediate from Theorem 4.9, since now $d_G(u) + d_G(v) \geq \nu_G$ for all $u, v \in V_G$ whether adjacent or not. □

Chvátal's condition

The Hamiltonian problem of graphs has attracted much attention, at least partly because the problem has practical significance. (Indeed, the first example where DNA computing was applied, was the Hamiltonian problem.)

There are some general improvements of the previous results of this chapter, and quite many improvements in various special cases, where the graphs are somehow restricted. We become satisfied by two general results.

For a graph G, let $V_G = \{v_1, v_2, \dots, v_n\}$ be ordered so that

$$d_G(v_1) \leq d_G(v_2) \leq \cdots \leq d_G(v_n)\,.$$

The (ascending) **degree sequence** of G is the sequence $d_G(v_1), d_G(v_2), \dots, d_G(v_n)$.

CHVÁTAL proved in 1972:

Theorem 4.11. *Let G be a graph of order $n \geq 3$ with the degree sequence $d_1, d_2, \dots, d_n$. If for every $i < n/2$,*

$$d_i \leq i \implies d_{n-i} \geq n - i\,, \tag{4.1}$$

then G is Hamiltonian.

Proof. First of all, we may suppose that G is closed, $G = cl(G)$, because G is Hamiltonian if and only if $cl(G)$ is Hamiltonian, and adding edges to G does not decrease any of its degrees, that is, if G satisfies (4.1), so does $G + e$ for every e. We show that, in this case, $G = K_n$, and thus G is Hamiltonian.

Assume on the contrary that $G \neq K_n$, and let $uv \notin E_G$ with $d_G(u) \leq d_G(v)$ be such that $d_G(u) + d_G(v)$ is as large as possible. Because G is closed, we must have $d_G(u) + d_G(v) < n$, and therefore $d_G(u) = i < n/2$. Let $A = \{w \mid vw \notin E_G, w \neq v\}$. By our choice, $d_G(w) \leq i$ for all $w \in A$, and, moreover,

$$|A| = (n-1) - d_G(v) \geq d_G(u) = i\,.$$

Consequently, there are at least i vertices w with $d_G(w) \leq i$, and so $d_i \leq d_G(u) = i$.

Similarly, for each vertex from $B = \{w \mid uw \notin E_G, w \neq u\}$, $d_G(w) \leq d_G(v) < n - d_G(u) = n - i$, and

$$|B| = (n-1) - d_G(u) = (n-1) - i\,.$$

Also $d_G(u) < n - i$, and thus there are at least $n - i$ vertices w with $d_G(w) < n - i$. Consequently, $d_{n-i} < n - i$. This contradicts the obtained bound $d_i \leq i$ and the condition (4.1). □

Note that the condition (4.1) is easily checkable for any given graph.

In the following result of CHVÁTAL AND ERDÖS (1972), $\alpha(G)$ denotes the maximum size of a stable subset of the graph G. We omit the proof of this result.

Theorem 4.12. *Let G be a graph of order at least 3. If $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian.*

5 Matchings

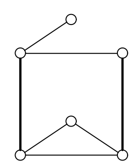
In matching problems we are given an availability relation between the elements of a set. The problem is then to find a pairing of the elements so that each element is paired (matched) uniquely with an available companion.

A special case of the matching problem is the **marriage problem**, which is stated as follows. Given a set X of boys and a set Y of girls, under what condition can each boy marry a girl who cares to marry him? This problem has many variations. One of them is the **job assignment problem**, where we are given n applicants and m jobs, and we should assign each applicant to a job he is qualified. The problem is that an applicant may be qualified for several jobs, and a job may be suited for several applicants.

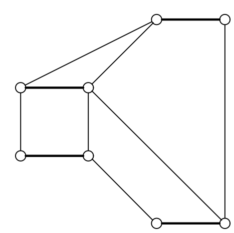
Maximum matchings

DEFINITION. For a graph G, a subset $M \subseteq E_G$ is a **matching** of G, if M contains no adjacent edges. The two ends of an edge $e \in M$ are **matched under** M. A matching M is a **maximum matching**, if for no matching M', $|M| < |M'|$. □

The two vertical edges on the right constitute a matching M that is *not a maximum matching*, although you cannot add any edges to M to form a larger matching. This matching is not maximum because the graph has a matching of three edges.



DEFINITION. A matching M **saturates** $v \in V_G$, if v is an end of an edge in M. Also, M **saturates** $A \subseteq V_G$, if it saturates every $v \in A$. If M saturates V_G, then M is a **perfect matching**. □

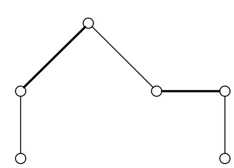


It is clear that every perfect matching is maximum.
On the right the horizontal edges form a perfect matching.

DEFINITION. Let M be a matching of G. An odd path $P = e_1 e_2 \dots e_{2k+1}$ is M-**augmented**, if

- P alternates between $E_G - M$ and M (that is, $e_{2i+1} \in E_G - M$ and $e_{2i} \in M$), and
- the ends of P are not saturated.

□



Lemma 5.1. *If G is connected with $\Delta(G) \le 2$, then G is a path or a cycle.*

Proof. Exercise. □

We start with a result of BERGE (1957) stating a necessary and sufficient condition for a matching to be maximal. One can use the first part of the proof to construct a maximum matching in an iterative manner starting from any matching M and from any M-augmented path.

Theorem 5.2. *A matching M of G is a maximum matching if and only if there are no M-augmented paths in G.*

Proof. ($\Rightarrow$) Let a matching M have an M-augmented path $P = e_1 e_2 \dots e_{2k+1}$ in G. Here $e_2, e_4, \dots, e_{2k} \in M$, $e_1, e_3, \dots, e_{2k+1} \notin M$. Define $N \subseteq E_G$ by

$$N = (M - \{e_{2i} \mid i \in [1,k]\}) \;\cup\; \{e_{2i+1} \mid i \in [0,k]\}\,.$$

Now, N is a matching of G, and $|N| = |M| + 1$. Therefore M is not a maximum matching.

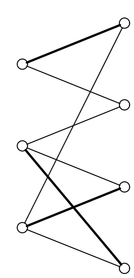
($\Leftarrow$) Assume N is a maximum matching, but M is not. Hence $|N| > |M|$. Consider the subgraph

$$H = G[M \bigtriangleup N], \quad \text{where } M \bigtriangleup N = (M - N) \cup (N - M)$$

is the symmetric difference of M and N. We have $d_H(v) \leq 2$ for each $v \in V_H$, because v is an end of at most one edge in M and N. By Lemma 5.1, each component A of H is either a path or a cycle.

Since no $v \in A$ can be an end of two edges from N or from M, each component (path or a cycle) A alternates between N and M. Now, since $|N| > |M|$, there is a component A of H, which has more edges from N than from M. This A cannot be a cycle, because an alternating cycle is even, and it thus contains equally many edges from N and M. Hence $A\colon u \xrightarrow{\star} v$ is a path, which starts and ends with an edge from N. Because A is a component of H, the ends u and v are not saturated by M, and, consequently, A is an M-augmented path. This proves the theorem. □

Hall's theorem



For a subset $S \subseteq V_G$ of a graph G, denote

$$N_G(S) = \{v \mid uv \in E_G \;\text{ for some }\; u \in S\}\,.$$

If G is (X,Y)-bipartite, and $S \subseteq X$, then $N_G(S) \subseteq Y$.

The following result, known as the **Marriage Theorem**, is due to HALL (1935).

Theorem 5.3. *Let G be a (X,Y)-bipartite graph. Then G contains a matching M saturating X if and only if*

(5.1) $$|S| \leq |N_G(S)| \quad \textit{for all} \;\; S \subseteq X.$$

Proof. (⇒) Let M be a matching that saturates X. If $|S| > |N_G(S)|$ for some $S \subseteq X$, then not all $x \in S$ can be matched with different $y \in N_G(S)$.

(⇐) Let G satisfy Hall's condition (5.1). We prove the claim by induction on $|X|$.

If $|X| = 1$, then the claim is clear. Let then $|X| \geq 2$, and assume (5.1) implies the existence of a matching that saturates every proper subset of X.

If $|N_G(S)| \geq |S| + 1$ for every nonempty $S \subseteq X$ with $S \neq X$, then choose an edge $uv \in E_G$ with $u \in X$, and consider the induced subgraph $H = G - \{u, v\}$. For all $S \subseteq X - \{u\}$,

$$|N_H(S)| \geq |N_G(S)| - 1 \geq |S|\,,$$

and hence, by the induction hypothesis, H contains a matching M saturating $X - \{u\}$. Now $M \cup \{uv\}$ is a matching saturating X in G, as was required.

Suppose then that there exists a nonempty subset $R \subseteq X$ with $R \neq X$ such that $|N_G(R)| = |R|$. The induced subgraph $H_1 = G[R \cup N_G(R)]$ satisfies (5.1) (since G does), and hence, by the induction hypothesis, H_1 contains a matching M_1 that saturates R (with the other ends in $N_G(R)$).

Also, the induced subgraph $H_2 = G[V_G - A]$, for $A = R \cup N_G(R)$, satisfies (5.1). Indeed, if there were a subset $S \subseteq X - R$ such that $|N_{H_2}(S)| < |S|$, then we would have

$$|N_G(S \cup R)| = |N_{H_2}(S)| + |N_{H_1}(R)| < |S| + |N_G(R)| = |S| + |R| = |S \cup R|$$

(since $S \cap R = \emptyset$), which contradicts (5.1) for G. By the induction hypothesis, H_2 has a matching M_2 that saturates $X - R$ (with the other ends in $Y - N_G(R)$). Combining the matchings for H_1 and H_2, we get a matching $M_1 \cup M_2$ saturating X in G. □

Corollary 5.4. *If G is a k-regular bipartite graph with $k > 0$, then G has a perfect matching.*

Proof. Let G be k-regular (X, Y)-bipartite graph. By regularity, $k \cdot |X| = \varepsilon_G = k \cdot |Y|$, and hence $|X| = |Y|$. Let $S \subseteq X$. Denote by E_1 the set of the edges with an end in S, and by E_2 the set of the edges with an end in $N_G(S)$. Clearly, $E_1 \subseteq E_2$. Therefore, $k \cdot |N_G(S)| = |E_2| \geq |E_1| = k \cdot |S|$, and so $|N_G(S)| \geq |S|$. By Theorem 5.3, G has a matching that saturates X. Since $|X| = |Y|$, this matching is necessarily perfect. □

Below we shall state Hall's theorem again in a different setting of transversals.

DEFINITION. Let $\mathcal{S} = \{S_1, S_2, \dots, S_m\}$ be a family of finite nonempty subsets of a set S. (These sets need not be distinct from each other.) A **transversal** (or a **system of distinct representatives**) of $\mathcal{S}$ is a subset $T \subseteq S$ of m distinct elements one from each S_i. □

As an example, let $S = [1, 6]$, and let

$$S_1 = S_2 = \{1, 2\},\ \ S_3 = \{2, 3\} \text{ and } S_4 = \{1, 4, 5, 6\}\,.$$

For $\mathcal{S} = \{S_1, S_2, S_3, S_4\}$, the set $T = \{1, 2, 3, 4\}$ is a transversal. If we add the set $S_5 = \{2, 3\}$ to $\mathcal{S}$, then it is impossible to find a transversal for this new family.

The connection of transversals to the Marriage Theorem is as follows. Let $S = Y$ and $X = [1, m]$. Form an (X, Y)-bipartite graph G such that there is an edge (i, s) if and only if $s \in S_i$. The possible transversals T of $\mathcal{S}$ are then obtained from the matchings M saturating X in G by taking the ends in Y of the edges of M.

Corollary 5.5. *Let $\mathcal{S}$ be a family of finite nonempty sets. Then $\mathcal{S}$ has a transversal if and only if the union of any k of the subsets S_i of $\mathcal{S}$ contains at least k elements.*

Tutte's theorem

The next theorem of Tutte (1947) is a classic characterization of perfect matchings.

Definition. A component of G is called **odd** (**even**), if it has an odd (even) number of vertices. Denote by $c_{\text{odd}}(G)$ the number of odd components in G. □

Theorem 5.6. *Let G be a nontrivial graph. The following are equivalent.*

(i) *G has a perfect matching.*
(ii) *For every proper subset $S \subset V_G$, $c_{\text{odd}}(G - S) \leq |S|$.*

Note that the condition in (ii) includes the case, where $S = \emptyset$.

Proof. (i)⇒(ii) Assume first that G has a perfect matching M, and let $S \subset V_G$. Now, for every odd component R of $G - S$, there is at least one edge of M from R to S, because no edge of G connects two different components of $G - S$, and M is perfect. This means that there are at least $c_{\text{odd}}(G - S)$ edges of M from $G - S$ to S. It follows that $c_{\text{odd}}(G - S) \leq |S|$, since otherwise there would be two edges of M with the same end in S contradicting the matching property of M.

(i)⇐(ii) Observe that *if G satisfies (ii), then so does every $G + e$.* Indeed, if addition of e merges two components of $G - S$, then the number of the odd components will not increase.

Assume the claim does not hold. There exists then a G satisfying (ii) such that

(a) G has no perfect matching,
(b) $G + e$ has a perfect matching for all $e \notin E_G$.

For $S = \emptyset$, $c_{\text{odd}}(G - S) \leq |S|$ implies that *ν_G is even.* Indeed, a graph of odd order has an odd component. Let

$$A = \{v \in V_G \mid d_G(v) = \nu_G - 1\}$$

be the set of those vertices adjacent to all other vertices.

(A) Assume first that the components of $G - A$ are all cliques, say $G_1, \dots, G_m$, where $G_1, \dots, G_t$ ($t \leq m$) have odd order. Now $t = c_{\text{odd}}(G - A) \leq |A|$. Let $X = \{u_1, \dots, u_t\} \subseteq A$, and let $G_{m+1} = G[A - X]$. Then G_{m+1} is a clique of even order, since ν_G is even.

Let M_i be a maximum matching of G_i for $i \in [1, m+1]$. Then M_i is perfect for all $i \in [t+1, m+1]$, and for each $i \in [1, t]$, there is exactly one vertex $v_i \in V_{G_i}$ not saturated by M_i. Now

$$M = \bigcup_{i=1}^{m+1} M_i \cup \{v_i u_i \mid i \in [1, t]\}$$

is a perfect matching of G; a contradiction.

(B) Therefore $G - A$ has a component that is not a clique. (Maybe $A = \emptyset$.)

In this case, there exist $u, v, w \in V_G$ in a component of $G - A$ such that $uw, vw \in E_G$ and $uv \notin E_G$. Since $w \notin A$, there exists a vertex z such that $wz \notin E_G$.

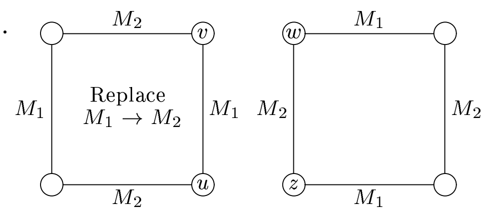
Consider $G + (uv)$ and its perfect matching M_1, and $G + (wz)$ and its perfect matching M_2 provided by (b). Let, as in the proof of Theorem 5.2, $F = M_1 \triangle M_2$, and let $H = (V_G, F)$. Clearly, $uv, wz \in F$. Each $v \in V_G$ is an end of exactly one edge from both M_1 and M_2, and so $d_H(v) = 0$ or 2 for all $v \in V_G$.

Therefore H consists of components that are even cycles, or isolated vertices. Moreover, each even cycle alternates between M_1 and M_2, since these are matchings.

Let C be the even cycle of H that contains uv. If wz is not on C, then

$$M = (M_1 - E_C) \cup (M_2 \cap E_C)$$

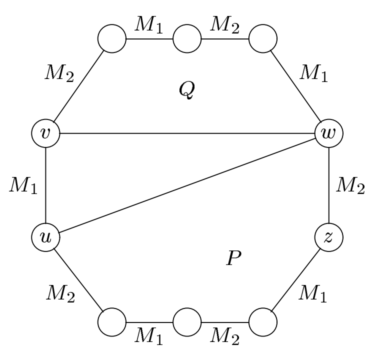
is a perfect matching of G; a contradiction.



Suppose then that C contains both uv and wz. Assume that

$$C = P(uv)Q \colon w \xrightarrow{\star} u \longrightarrow v \xrightarrow{\star} w\,,$$

where $P = (wz)P'$. (The symmetric case, where v comes before u in C, is treated similarly.) The path $P \colon w \xrightarrow{\star} u$ is odd (because it starts and ends with an edge from M_2). Since $|C|$ is even, $|Q|$ is even.



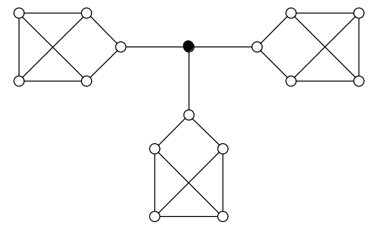
Consider

$$M = (M_1 - E_C) \cup \{uw\} \cup (M_1 \cap E_P) \cup (M_2 \cap E_Q)\,.$$

This is a perfect matching of G; a contradiction. □

Tutte's theorem does not provide a good algorithm for constructing a perfect matching, because the theorem requires 'too many cases'. Its applications are mainly in the proofs of other results that are related to matchings. There is a good algorithm due to EDMONDS (1965), which uses 'blossom shrinkings', but this algorithm is somewhat involved.

The next 3-regular graph does not have a perfect matching, because removing the black vertex results in a graph with three odd components.



Using Theorem 5.6 we can give a short proof of PETERSEN's result for 3-regular graphs (1891).

Theorem 5.7. *If G is a bridgeless 3-regular graph, then it has a perfect matching.*

Proof. Let S be a proper subset of V_G, and let G_i, $i \in [1,t]$, be the odd components of $G-S$. Denote by m_i the number of edges with one end in G_i and the other in S. Since G is 3-regular,

$$\sum_{v \in V_{G_i}} d_G(v) = 3 \cdot \nu_{G_i} \quad \text{and} \quad \sum_{v \in S} d_G(v) = 3 \cdot |S| .$$

The first of these implies that

$$m_i = \sum_{v \in v_{G_i}} d_G(v) - 2 \cdot \varepsilon_{G_i}$$

is odd. Furthermore, $m_i \neq 1$, because G has no bridges, and therefore $m_i \geq 3$. Hence the number of odd components of $G-S$ satisfies

$$t \leq \frac{1}{3} \sum_{i=1}^{t} m_i \leq \frac{1}{3} \sum_{v \in S} d_G(v) = |S| ,$$

and so, by Theorem 5.6, G has a perfect matching. □

6 Edge Colourings

Colourings of edges and vertices of a graph G are useful, when one is interested in classifying relations between objects.

There are two sides of colourings. In the general case, a graph G with a colouring α is given, and we study the properties of this pair $G^\alpha = (G, \alpha)$. This is the situation, *e.g.*, in transportation networks with bus and train links, where the colour (*buss, train*) of an edge tells the nature of a link.

In the chromatic theory, G is first given and then we search for a colouring that the satisfies required properties. One of the important properties of colourings is 'properness'. In a proper colouring adjacent edges or vertices are coloured differently.

Edge chromatic number

DEFINITION. A k-**edge colouring** $\alpha\colon E_G \to [1,k]$ of a graph G is an assignment of k colours to its edges. We write G^α to indicate that G has the edge colouring α.

A vertex $v \in V_G$ and a colour $i \in [1,k]$ are **incident** with each other, if $\alpha(vu) = i$ for some $vu \in E_G$. If $v \in V_G$ is not incident with a colour i, then i is **available** for v.

The colouring α is **proper**, if no two adjacent edges obtain the same colour: $\alpha(e_1) \neq \alpha(e_2)$ for adjacent e_1 and e_2.

The **edge chromatic number** $\chi'(G)$ of G is defined as

$$\chi'(G) = \min\{k \mid \text{ there exists a proper } k\text{-edge colouring of } G\}.$$

□

A k-edge colouring α can be thought of as a partition $\{E_1, E_2, \dots, E_k\}$ of E_G, where $E_i = \{e \mid \alpha(e) = i\}$. Note that it is possible that $E_i = \emptyset$ for some i. We adopt a simplified notation

$$G^\alpha[i_1, i_2, \dots, i_t]$$

for the subgraph $G[E_{i_1} \cup E_{i_2} \cup \dots \cup E_{i_t}] \subseteq G$ consisting of those edges that have a colour i_1, i_2, ... , or i_t. That is, the edges having other colours are removed.

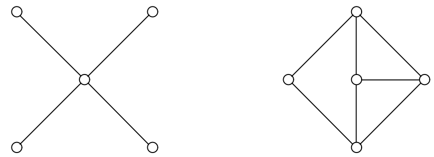
Lemma 6.1. *Each colour set E_i in a proper k-edge colouring is a matching.*

Proof. This is clear. □

Lemma 6.2. $\Delta(G) \leq \chi'(G) \leq \varepsilon_G.$

Proof. This is clear. □

The three numbers in Lemma 6.2 can be equal. This happens, for instance, when $G = K_{1,n}$ is a star. But often the inequalities are strict.



A star, and a graph with $\chi'(G) = 4$.

Optimal colourings

We show that for bipartite graphs the lower bound is always optimal: $\chi'(G) = \Delta(G)$.

Lemma 6.3. *Let G be a connected graph that is not an odd cycle. Then there exists a 2-edge colouring (that need not be proper), in which both colours are incident with each vertex v with $d_G(v) \geq 2$.*

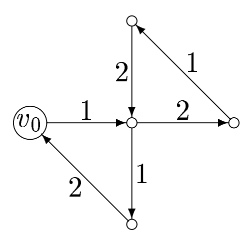
Proof. Assume that G is nontrivial; otherwise, the claim is trivial.

(1) Suppose first that G is Eulerian. If G is an even cycle, then a 2-edge colouring exists as required. Otherwise, since now $d_G(v)$ is even for all v, G has a vertex v_1 with $d_G(v_1) \geq 4$. Let $e_1 e_2 \dots e_t$ be an Euler tour of G, where $e_i = v_i v_{i+1}$ (and $v_{t+1} = v_1$). Define

$$\alpha(e_i) = \begin{cases} 1, & \text{if } i \text{ is odd}\,, \\ 2, & \text{if } i \text{ is even}\,. \end{cases}$$

Hence the ends of the edges e_i for $i \in [2, t-1]$ are incident with both colours. All vertices are among these ends. The condition $d_G(v_1) \geq 4$ guarantees this for v_1. Hence the claim holds in the Eulerian case.

(2) Suppose then that G is not Eulerian. We define a new graph G_0 by adding a vertex v_0 to G and connecting v_0 to each $v \in V_G$ of odd degree.
In G_0 every vertex has even degree including v_0 (by the handshaking lemma), and hence G_0 is Eulerian. Let $e_0 e_1 \dots e_t$ be an Eulerian tour of G_0, where $e_i = v_i v_{i+1}$. By the previous case, there is a required colouring α of G_0. Now, α restricted to E_G is clearly a colouring of G as required by the claim.



□

DEFINITION. For a k-edge colouring α of G, let

$$c_\alpha(v) = |\{i \mid v \text{ is incident with } i \in [1, k]\}|\,.$$

A k-edge colouring β is an **improvement** of α, if

$$\sum_{v \in V_G} c_\beta(v) > \sum_{v \in V_G} c_\alpha(v)\,.$$

Also, α is **optimal**, if it cannot be improved. □

Notice that we always have $c_\alpha(v) \le d_G(v)$, and if α is proper, then $c_\alpha(v) = d_G(v)$, and in this case α is optimal. Thus an improvement of a colouring is a change towards a proper colouring. Note also that a graph G always has an optimal k-edge colouring, but it need not have any proper k-edge colourings.

The next lemma is obvious.

Lemma 6.4. *An edge colouring α of G is proper if and only if $c_\alpha(v) = d_G(v)$ for all vertices $v \in V_G$.*

Lemma 6.5. *Let α be an optimal k-edge colouring of G, and let $v \in V_G$. Suppose that the colour i is available for v, and the colour j is incident with v at least twice. Then the component H of $G^\alpha[i,j]$ that contains v, is an odd cycle.*

Proof. Suppose the component H is not an odd cycle. By Lemma 6.3, H has a 2-edge colouring γ, in which both colours are incident with each vertex of degree at least two. We may suppose (by recycling the colours if necessary) that the two colours in γ are i and j. In this way we obtain a new colouring β of G,

$$\beta(e) = \begin{cases} \gamma(e), & \text{if } e \in E_H\,, \\ \alpha(e), & \text{if } e \notin E_H\,. \end{cases}$$

Since $d_H(v) \ge 2$ (by the assumption on the colour j) and in β both colours i and j are now incident with v, $c_\beta(v) = c_\alpha(v) + 1$. Furthermore, by the construction of β, we have $c_\beta(u) \ge c_\alpha(u)$ for all $u \ne v$. Therefore $\sum_{u \in V_G} c_\beta(u) > \sum_{u \in V_G} c_\alpha(u)$, which contradicts the optimality of α. Hence H is an odd cycle. □

The next theorem is due to König (1916).

Theorem 6.6. *If G is bipartite, then $\chi'(G) = \Delta(G)$.*

Proof. Let α be an optimal Δ-edge colouring of a bipartite G, where $\Delta = \Delta(G)$. If there were a $v \in V_G$ with $c_\alpha(v) < d_G(v)$, then by Lemma 6.5, G would contain an odd cycle. But a bipartite graph does not contain such cycles. Therefore, for all vertices v, $c_\alpha(v) = d_G(v)$. By Lemma 6.4, α is a proper colouring, and $\Delta = \chi'(G)$ as required. □

Vizing's theorem

In general we can have $\chi'(G) > \Delta(G)$ as one of our examples did show. The following important theorem, due to Vizing (1964), shows that the edge chromatic number of a graph G misses $\Delta(G)$ by at most one colour.

Theorem 6.7. *For any graph G, $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$.*

Proof. We need only to show that $\chi'(G) \le \Delta(G) + 1$. Suppose on the contrary that $\chi'(G) > \Delta(G) + 1$, and let α be an optimal $(\Delta + 1)$-edge colouring of G.

We have (trivially) $d_G(u) < \Delta + 1 < \chi'(G)$ for all $u \in V_G$, and so

Claim 1. *For each $u \in V_G$, there exists an available colour $b(u)$ for u.*

Since, by the counter hypothesis, α is not a proper colouring, there exists a $v \in V_G$ with $c_\alpha(v) < d_G(v)$, and hence a colour i_1 that is incident with v at least twice, say

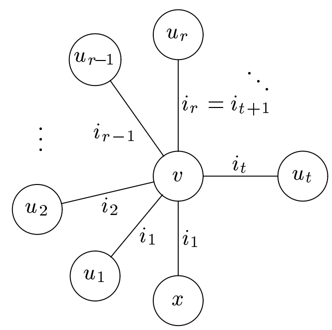
$$\alpha(vu_1) = i_1 = \alpha(vx) .$$

We define inductively a sequence $u_1, u_2, \dots$ of vertices as follows:

$$\alpha(vu_j) = i_j \quad \text{and} \quad i_{j+1} = b(u_j) .$$

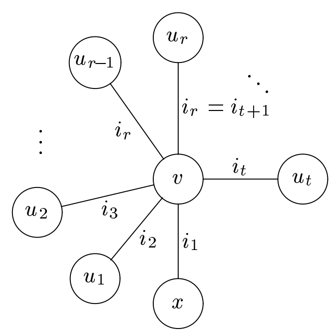
This is a well defined sequence by Claim 1, and by the following inductive claim.

Claim 2. *The vertex v is incident with each colour i_{j+1}: $\alpha(vu_{j+1}) = i_{j+1}$.*



Indeed, otherwise we can recolour the edges vu_ℓ by $i_{\ell+1}$ for $\ell \in [1, j]$, and obtain an improvement of α. (Here v gains a new colour, and for each u_ℓ either its number of colours remains the same or it increases by one.)

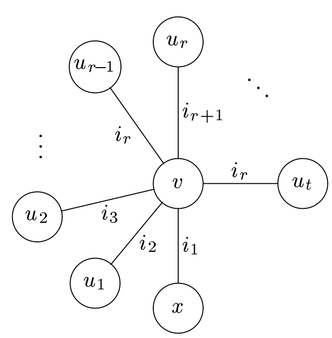
Let t be the smallest index such that for some $r < t$, $i_{t+1} = i_r$. Such an index t exists, because $d_G(v)$ is finite.



Let β be a recolouring of G such that for $1 \le j \le r-1$, $\beta(vu_j) = i_{j+1}$, and for all other edges e, $\beta(e) = \alpha(e)$.

Claim 3. *β is an optimal $(\Delta + 1)$-edge colouring of G.*

Indeed, $c_\beta(v) = c_\alpha(v)$ and $c_\beta(u) \ge c_\alpha(u)$ for all u, since each u_j $(1 \le j \le r-1)$ gains a new colour j_{i+1} although it may loose one of its old colours.



Let then the colouring γ be obtained from β by recolouring the edges vu_j by i_{j+1} for $r \le j \le t$. Now, vu_t is recoloured by $i_r = i_{t+1}$.

Claim 4. *γ is an optimal $(\Delta + 1)$-edge colouring of G.*

Indeed, the fact $i_r = i_{t+1}$ ensures that i_r is a new colour incident with u_t, and thus that $c_\gamma(u_t) \ge c_\beta(u_t)$. For all other vertices, $c_\gamma(u) \ge c_\beta(u)$ follows as for β.

Let $i_0 = b(v)$ be available for v, see the Claim 1. By Lemma 6.5, the components H_1 of $G^\beta[i_0, i_r]$ and H_2 of $G^\gamma[i_0, i_r]$ containing the vertex v are cycles, that is, H_1 is a cycle $(vu_{r-1})P_1(u_r v)$ and H_2 is a cycle $(vu_{r-1})P_2(u_t v)$, where $P_1 \colon u_{r-1} \xrightarrow{\star} u_r$ and $P_2 \colon u_{r-1} \xrightarrow{\star} u_t$ are paths. However, the edges of P_1 and P_2 have the same colours with respect to β and γ (either i_0 or i_r). This is not possible, since P_1 ends in u_r while P_2 ends in a different vertex u_t. This contradiction proves the theorem. □

Research problem: Vizing's theorem (nor its present proof) does not offer any characterization for the graphs, for which $\chi'(G) = \Delta(G) + 1$. In fact, it is one of the famous open problems of graph theory to find such a characterization. The answer is known (only) for some special classes of graphs. By Holyer (1981), the problem whether $\chi'(G)$ is $\Delta(G)$ or $\Delta(G) + 1$ is NP-complete.

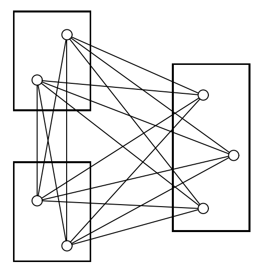
The proof of Vizing's theorem can be used to obtain a proper colouring of G with at most $\Delta(G) + 1$ colours, when the word 'optimal' is forgotten: colour first the edges as well as you can (if nothing better, then arbitrarily in two colours), and use the proof iteratively to improve the colouring until no improvement is possible – then the proof says that the result is a proper colouring.

7 Ramsey Theory

In general, Ramsey theory studies unavoidabilities of combinatorial properties. We consider an instance of this theory mainly for edge colourings (that need not be proper). A typical example of a Ramsey property is the following: given 6 persons each pair of whom are either friends or enemies, there are then 3 persons who are mutual friends or mutual enemies. In graph theoretic terms this means that each colouring of the edges of a clique K_6 with 2 colours results in a monochromatic triangle.

Turan's theorem

We shall first consider the problem of finding a general condition for a clique K_p to appear in a graph. It is clear that every graph contains K_1, and that every nondiscrete graph contains K_2.



DEFINITION. A **complete p-partite graph** G consists of p discrete and disjoint induced subgraphs $G_1, G_2, \dots, G_p$, where $uv \in E_G$ if and only if u and v belong to different parts, G_i and G_j with $i \neq j$. □

Note that a complete p-partite graph is completely determined by its discrete parts G_i, $i \in [1, p]$.

Let $p \geq 3$, and let H be the complete $(p-1)$-partite graph of order $n = t(p-1)+r$, where $r \in [1, p-1]$ and $t \geq 0$, such that there are r parts $H_1, \dots, H_r$ of order $t+1$ and $p-1-r$ parts $H_{r+1}, \dots, H_{p-1}$ of order t (when $t > 0$). (Here r is the positive residue of n modulo $(p-1)$, and is thus determined by n and p.)

By its definition, H does not have any cliques K_p as a subgraph. One can compute that the number ε_H of edges of H is equal to

$$T(n,p) = \frac{p-2}{2(p-1)}n^2 - \frac{r}{2}\left(1 - \frac{r}{p-1}\right). \tag{7.1}$$

The next TURÁN's result (1941) shows that the above bound $T(n,p)$ is optimal.

Theorem 7.1. *If a graph G of order n has $\varepsilon_G > T(n,p)$ edges, then G contains a clique K_p.*

Proof. Let $n = (p-1)t + r$ for $1 \leq r \leq p-1$ and $t \geq 0$. We prove the claim by induction on t. If $t = 0$, then G has no K_p.

Suppose then that $t \geq 1$, and let G be a graph of order n such that ε_G is maximum subject to the condition $K_p \nsubseteq G$.

Now G contains a clique $G[A] = K_{p-1}$, since adding any one edge to G results in a K_p, and $p-1$ vertices of this K_p induce a subgraph K_{p-1} of G.

Each $v \notin A$ is adjacent to at most $p-2$ vertices of A; otherwise $G[A \cup \{v\}] = K_p$. Furthermore, $K_p \nsubseteq G{-}A$, and $\nu_{G[A\cup\{v\}]} = n{-}p{+}1$. Because $n{-}p{+}1 = (t{-}1)(p{-}1){+}r$, we can apply the induction hypothesis to obtain $\varepsilon_{G-A} \le T(n-p+1, p)$. Now

$$\varepsilon_G \le T(n-p+1,p) + (n-p+1)(p-2) + \frac{(p-1)(p-2)}{2} = T(n,p)\,,$$

which proves the claim. □

Triangles in graphs

When Theorem 7.1 is applied to triangles K_3, we have a result of Mantel (1907).

Corollary 7.2. *If a graph G of order n has $\varepsilon_G > \frac{1}{4}n^2$ edges, then G contains a triangle, $K_3 \subseteq G$.*

Definition. Let α be an edge colouring of G. A subgraph $H \subseteq G$ is said to be (i-)**monochromatic**, if all edges of H have the same colour i. A monochromatic subgraph K_3 is a **monochromatic triangle** of G. The number of monochromatic triangles in G is denoted by $t(G^\alpha)$. □

Theorem 7.3. *Let α be a 2-edge colouring of K_n, and for each vertex v, define $r_v = |\{e \mid e = vu,\ \alpha(e) = 1\}|$. Then*

$$t(K_n^\alpha) = \binom{n}{3} - \frac{1}{2}\sum_{v\in V} r_v(n-1-r_v). \tag{7.2}$$

Proof. There are $\binom{n}{3}$ triangles in K_n. Every triangle that is not monochromatic has exactly two vertices incident with both colours 1 and 2. For a vertex v, the number of pairs of edges (vu_1, vu_2) of different colour is $r_v(n-1-r_v)$, because $n-1-r_v$ is the number of edges vu with $\alpha(vu) = 2$. This means that each polychromatic triangle is counted twice in $\sum_{v\in V} r_v(n-1-r_v)$, and hence the claim. □

We can calculate a lower bound for $t(G^\alpha)$ from (7.2). Omitting the details we mention:

$$t(K_n^\alpha) \ge \binom{n}{3} - \left\lfloor \frac{n}{2} \left\lfloor \left(\frac{n-1}{2}\right)^2 \right\rfloor \right\rfloor .$$

This formula gives in the case $n = 6$ a lower bound $t(K_6^\alpha) \ge 2$, and so no matter how you colour the edges of a clique K_6 with red and blue, there are always at least two monochromatic triangles.

Ramsey's theorem

The generalization of the above result for triangles was provided by RAMSEY (1930).

Theorem 7.4. *Let $p, q \geq 2$ be any integers. Then there exists a (smallest) integer $R(p,q)$ such that for all $n \geq R(p,q)$, any 2-edge colouring of $K_n \to [1,2]$ contains a 1-monochromatic K_p or a 2-monochromatic K_q.*

Before proving this, we give an equivalent statement. Recall that a subset $X \subseteq V_G$ is stable, if $G[X]$ is a discrete graph.

Theorem 7.5. *Let $p, q \geq 2$ be any integers. Then there exists a (smallest) integer $R(p,q)$ such that for all $n \geq R(p,q)$, any graph G of order n contains a clique of order p or a stable set of order q.*

Be patient, this will follow from Theorem 7.7. The number $R(p,q)$ is known as the **Ramsey number** for p and q.

Example 7.6. It is clear that $R(p,2) = p$ and $R(2,q) = q$. □

Theorems 7.4 and 7.5 follow from the next result of ERDÖS AND SZEKERES (1935).

Theorem 7.7. *The Ramsey number $R(p,q)$ exists for all $p, q \geq 2$, and*

$$R(p,q) \leq R(p,q-1) + R(p-1,q) .$$

Proof. We use induction on $p+q$. By Example 7.6, $R(p,q)$ exists for $p = 2$ or $q = 2$, and it is thus exists for $p + q \leq 5$.

It is now sufficient to show that if G is a graph of order $R(p,q-1) + R(p-1,q)$, then it has a clique of order p or a stable subset of order q.

Let $v \in V_G$, and denote by $A = V_G - (N_G(v) \cup \{v\})$ the set of vertices that are not adjacent to v. Since G has $R(p,q-1) + R(p-1,q) - 1$ vertices different from v, either $|N_G(v)| \geq R(p-1,q)$ or $|A| \geq R(p,q-1)$ (or both).

Assume first that $|N_G(v)| \geq R(p-1,q)$. By the definition of Ramsey numbers, $G[N_G(v)]$ contains a clique B of order $p-1$ or a stable subset S of order q. In the first case, $B \cup \{v\}$ induces a clique K_p in G, and in the second case the same stable set of order q is good for G.

If $|A| \geq R(p,q-1)$, then $G[A]$ contains a clique of order p or a stable subset S of order $q-1$. In the first case, the same clique of order p is good for G, and in the second case, $S \cup \{v\}$ is a stable subset of G of q vertices. This proves the claim. □

The following was also shown by Erdös and Szekeres:

Theorem 7.8. *For all $p, q \geq 2$,*

$$R(p,q) \leq \binom{p+q-2}{p-1}.$$

Proof. For $p = 2$ or $q = 2$, the claim is clear from Example 7.6. We use induction on $p + q$ for the general statement. Assume that $p, q \geq 3$. By Theorem 7.7 and the induction hypothesis,

$$\begin{aligned} R(p,q) &\leq R(p,q-1) + R(p-1,q) \\ &\leq \binom{p+q-3}{p-1} + \binom{p+q-3}{p-2} = \binom{p+q-2}{p-1}, \end{aligned}$$

which is what we wanted. □

In the table below we give some known values and estimates for the Ramsey numbers $R(p,q)$. As can be read from the table, not much is known about these numbers.

$p \backslash q$	3	4	5	6	7	8	9	10
3	6	9	14	18	23	28	36	40-43
4	9	18	25	35-41	49-61	55-84	69-115	80-149
5	14	25	43-49	58-87	80-143	95-216	116-316	118-442

The first unknown $R(p,p)$ (where $p = q$) is for $p = 5$. It has been verified that $43 \leq R(5,5) \leq 49$, but to determine the exact value is an open problem.

Generalizations*

Theorem 7.4 can be generalized as follows.

Theorem 7.9. *Let $q_i \geq 2$ be integers for $i \in [1,k]$ with $k \geq 2$. Then there exists an integer $R = R(q_1, q_2, \dots, q_k)$ such that for all $n \geq R$, any k-edge colouring of K_n has an i-monochromatic K_{q_i} for some i.*

Proof. The proof is by induction on k. The case $k = 2$ is treated in Theorem 7.4. For $k > 2$, we show that $R(q_1, \dots, q_k) \leq R(q_1, \dots, q_{k-2}, p)$, where $p = R(q_{k-1}, q_k)$.

Let $n = R(q_1, \dots, q_{k-2}, p)$, and let $\alpha \colon E_{K_n} \to [1,k]$ be an edge colouring. Let $\beta \colon E_{K_n} \to [1, k-1]$ be obtained from α by identifying the colours $k-1$ and k:

$$\beta(e) = \begin{cases} \alpha(e) & \text{if } \alpha(e) < k-1\,, \\ k-1 & \text{if } \alpha(e) = k-1 \text{ or } k\,. \end{cases}$$

By the induction hypothesis, K_n^β has an i-monochromatic K_{q_i} for some $1 \le i \le k-2$ (and we are done, since this subgraph is monochromatic in K_n^α) or K_n^β has a $(k-1)$-monochromatic subgraph $H^\beta = K_p$. In the latter case, by Theorem 7.4, H^α and thus K_n^α has a $(k-1)$-monochromatic or a k-monochromatic subgraph, and this proves the claim. □

Since each graph H is a subgraph of a clique K_m for $m = \nu_H$, we have

Corollary 7.10. *Let $k \ge 2$ and $H_1, H_2, \ldots, H_k$ be arbitrary graphs. Then there exists an integer $R(H_1, H_2, \ldots, H_k)$ such that for all cliques K_n with $n \ge R(H_1, H_2, \ldots, H_k)$ and for all k-edge colourings α of K_n, K_n^α contains an i-monochromatic subgraph H_i for some i.*

This generalization is trivial from Theorem 7.9. However, the generalized Ramsey numbers $R(H_1, H_2, \ldots, H_k)$ can be much smaller than their counter parts (for cliques) in Theorem 7.9.

Example 7.11. We state as an exercise:

Let T be a tree of order m. Then

$$R(T, K_n) = (m-1)(n-1) + 1\,,$$

that is, any graph G of order at least $R(T, K_n)$ contains a subgraph isomorphic to T, or the complement of G contains a clique K_n. □

It follows from Theorem 7.4 that for any clique K_n, there *exists* a graph G (well, any sufficiently large clique) such that any 2-edge colouring of G has a monochromatic (induced) subgraph K_n. Note, however, that in Corollary 7.10 the monochromatic subgraph H_i is not required to be induced.

The following impressive theorem, which improves the results we have mentioned in this chapter and which has a difficult proof, was shown to hold by Deuber, and Erdös, Hajnal and Pósa, and Rödl around 1973.

Theorem 7.12. *Let H be any graph. Then there exists a graph G such that any 2-edge colouring of G has an monochromatic induced subgraph H.*

Example 7.13. As an application of Ramsey's theorem, we shortly describe Schur's theorem. For this, consider the partition $\{1, 4, 10, 13\}$, $\{2, 3, 11, 12\}$, $\{5, 6, 7, 8, 9\}$ of the set $\mathbb{N}_{13} = [1, 13]$. We observe that in no partition class there are three integers such that $x + y = z$. However, if you try to partition $\mathbb{N}_{14}$ into three classes, then you are bound to find a class, where $x + y = z$ has a solution.

SCHUR (1916) solved this problem in a general setting. The following gives a short proof using Ramsey's theorem.

For each $n \geq 1$, there exists an integer $S(n)$ such that any partition $S_1, \dots, S_n$ of $\mathbb{N}_{S(n)}$ has a class S_i containing two integers x, y such that $x + y \in S_i$.

Indeed, let $S(n) = R(3, 3, \dots, 3)$, where 3 occurs n times, and let K be a clique on $\mathbb{N}_{S(n)}$. For a partition $S_1, \dots, S_n$ of $\mathbb{N}_{S(n)}$, define an edge colouring α of K by

$$\alpha(ij) = k, \text{ if } |i - j| \in S_k .$$

By Theorem 7.9, K^α has a monochromatic triangle, that is, there are three vertices $1 \leq i < j < t \leq S(n)$ such that $t-j, j-i, t-i \in S_k$ for some k. But $(t-j)+(j-i) = t-i$ proves the claim. □

There are quite many interesting corollaries to Ramsey's theorem in various parts of mathematics including not only graph theory, but also, *e.g.*, geometry and algebra, see

R.L. GRAHAM, B.L. ROTHSCHILD AND J.L. SPENCER, "Ramsey Theory", Wiley, (2nd ed.) 1990.

8 Vertex Colourings

The vertices of a graph G can also be classified using colourings. These colourings tell that certain vertices have a common property (or that they are similar in some respect), if they share the same colour. In this chapter, we shall concentrate on proper vertex colourings, where adjacent vertices get different colours.

The chromatic number

Definition. A k-**colouring** (or a k-**vertex colouring**) of a graph G is a mapping $\alpha\colon V_G \to [1,k]$. The colouring α is **proper**, if adjacent vertices obtain a different colour: for all $uv \in E_G$, we have $\alpha(u) \neq \alpha(v)$. A colour $i \in [1,k]$ is said to be **available** for a vertex v, if no neighbour of v is coloured by i.

The (vertex) **chromatic number** $\chi(G)$ of G is defines as

$$\chi(G) = \min\{k \mid \text{ there exists a proper } k\text{-colouring of } G\}\,.$$

If $\chi(G) = k$, then G is k-**chromatic**. □

Each proper vertex colouring $\alpha\colon V_G \to [1,k]$ provides a partition $\{V_1, V_2, \dots, V_k\}$ of the vertex set V_G, where

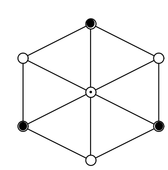
$$V_i = \{v \mid \alpha(v) = i\}\,.$$

We adopt the notation

$$G^\alpha[i_1, i_2, \dots, i_t]$$

for the induced subgraph $G[V_{i_1} \cup V_{i_2} \cup \dots \cup V_{i_t}]$ on the vertices of colours $\{i_1, \dots, i_t\}$.

The graph on the right, which is often called a wheel (of order 7), is 3-chromatic.



Example 8.1. By the definitions, a graph G is 2-chromatic if and only if it is bipartite. □

Lemma 8.2. *Let α be a proper k-colouring of G, and let π be any permutation of the colours. Then the colouring $\beta = \pi\alpha$ is a proper k-colouring of G.*

Proof. Indeed, if $\alpha\colon V_G \to [1,k]$ is proper, and if $\pi\colon [1,k] \to [1,k]$ is a bijection, then $uv \in E_G$ implies that $\alpha(u) \neq \alpha(v)$, and hence also that $\pi\alpha(u) \neq \pi\alpha(v)$. It follows that $\pi\alpha$ is a proper colouring. □

Critical graphs

DEFINITION. A k-chromatic graph G is said to be k-**critical**, if $\chi(H) < k$ for all proper subgraphs $H \subset G$. □

In a critical graph an elimination of any edge and of any vertex will reduce the chromatic number: $\chi(G-e) < \chi(G)$ and $\chi(G-v) < \chi(G)$ for $e \in E_G$ and $v \in V_G$. Each K_n is n-critical, since in $K_n-(uv)$ the vertices u and v can gain the same colour.

Theorem 8.3. *If G is k-critical for $k \geq 2$, then it is connected, and $\delta(G) \geq k-1$.*

Proof. Note that for any graph G with the components $G_1, G_2, \ldots, G_m$, $\chi(G) = \max\{\chi(G_i) \mid i \in [1,m]\}$. Connectivity claim follows from this observation.

Let then G be k-critical, but $\delta(G) = d_G(v) \leq k-2$ for $v \in V_G$. Since G is critical, there is a proper $(k-1)$-colouring of $G-v$. Now v is adjacent to only $\delta(G) < k-1$ vertices. But there are k colours, and hence there is an available colour i for v. If we recolour v by i, then a proper $(k-1)$-colouring is obtained for G; a contradiction. □

The case (iv) of the next theorem is the SZEKERES AND WILF's theorem (1968).

Theorem 8.4. *Let G be any graph with $k = \chi(G)$.*

(i) *G has a k-critical subgraph H.*
(ii) *G has at least k vertices of degree $\geq k-1$.*
(iii) $k \leq \Delta(G)+1$
(iv) $k \leq 1 + \max_{H \subseteq G} \delta(H)$.

Proof. For (i), we observe that a k-critical subgraph $H \subseteq G$ is obtained by removing vertices and edges from G as long as the chromatic number remains k.

For (ii), let $H \subseteq G$ be a k-critical subgraph. By Theorem 8.3, $d_H(v) \geq k-1$ for every $v \in V_H$. Of course, also $d_G(v) \geq k-1$ for every $v \in V_H$. The claim follows, because, clearly, every k-critical graph H must have at least k vertices.

For (iii), we use *greedy colouring*. Let $V_G = \{v_1, \ldots, v_n\}$ be ordered in some way, and define $\alpha\colon V_G \to \mathbb{N}$ inductively as follows: $\alpha(v_1) = 1$, and

$$\alpha(v_i) = \min\{j \mid \alpha(v_t) \neq j \text{ for all } t < j \text{ with } v_j v_t \in E_G\}.$$

Then α is proper, and $\alpha(v_i) \leq d_G(v_i) + 1$ for all i. The claim follows.

For (iv), let $H \subseteq G$ be a k-critical subgraph. By Theorem 8.3, $\chi(G) - 1 \leq \delta(H)$, which proves this claim. □

Lemma 8.5. *Let v be a cut vertex of a connected G, and let A_i, for $i \in [1,m]$, be the components of $G-v$. Denote $G_i = G[A_i \cup \{v\}]$. Then $\chi(G) = \max\{\chi(G_i) \mid i \in [1,m]\}$. In particular, a critical graph does not have cut vertices.*

Proof. Suppose that each component G_i has a proper k-colouring α_i. By Lemma 8.2, we may take $\alpha_i(v) = 1$ for all i. These k-colourings give a k-colouring of G. □

Brooks' theorem

For *edge* colourings we have a strong result, Vizing's theorem, but no such strong results are known for vertex colourings. Although, as we have seen, we always have $\chi(G) \leq \Delta(G) + 1$, the chromatic number $\chi(G)$ usually takes much lower values – as seen in the bipartite case. Moreover, the maximum value $\Delta(G) + 1$ is obtained only in two special cases as was shown by BROOKS in 1941.

The next proof of Brook's theorem is by LOVÁSZ (1975) as modified by BRYANT (1996).

Lemma 8.6. *Let G be a 2-connected graph. Then the following are equivalent:*

(i) *G is a clique or a cycle.*
(ii) *For all $u, v \in V_G$, if $uv \notin E_G$, then $\{u, v\}$ is a separating set.*
(iii) *For all $u, v \in V_G$, if $d_G(u, v) = 2$, then $\{u, v\}$ is a separating set.*

Proof. It is clear that (i) implies (ii), and that (ii) implies (iii). We need only to show that (iii) implies (i). Assume then that (iii) holds.

We shall show that either G is a clique or $d_G(v) = 2$ for all $v \in V_G$, from which the theorem follows.

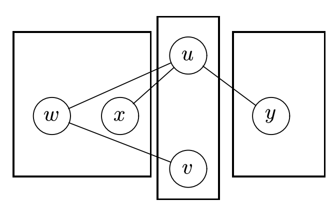
First of all, $d_G(v) \geq 2$ for all v, since G is 2-connected. Let w be a vertex of maximum degree, $d_G(w) = \Delta(G)$.

If the neighbourhood $N_G(w)$ is a clique, then G is a clique. Indeed, otherwise, since G is connected, there exists a vertex $u \notin N_G(w) \cup \{w\}$ such that u is adjacent to a vertex $v \in N_G(w)$. But then $d_G(v) > d_G(w)$, and this contradicts the choice of w.

Assume then that there are different vertices $u, v \in N_G(w)$ such that $uv \notin E_G$. This means that $d_G(u, v) = 2$ (the shortest path is $u \longrightarrow w \longrightarrow v$), and by (iii), $\{u, v\}$ is a separating set of G. Consequently, there is a partition $V_G = W \cup \{u, v\} \cup U$, where $w \in W$, and all paths from a vertex of W to a vertex of U go through either u or v.

We claim that $W = \{w\}$, and thus that $\Delta(G) = 2$ as required. Suppose on the contrary that $|W| \geq 2$. Since w is not a cut vertex (since G has no cut vertices), there exists an $x \in W$ with $x \neq w$ such that $xu \in E_G$ or $xv \in E_G$, say $xu \in E_G$.

Since v is not a cut vertex, there exists a $y \in U$ such that $uy \in E_G$. Hence $d_G(x, y) = 2$, and by (iii), $\{x, y\}$ is a separating set. Accordingly, $V_G = W_1 \cup \{x, y\} \cup U_1$, where all paths from W_1 to U_1 pass through x or y. Assume that $w \in W_1$, and hence that also $u, v \in W_1$. (Since $uw, vw \in E_{V_G - \{x,y\}}$).



There exists a vertex $z \in U_1$. Note that $U_1 \subseteq W \cup U$. If $z \in W$ (or $z \in U$, respectively), then all paths from z to u must pass through x (or y, respectively), and x (or y, respectively) would be a cut vertex of G. This contradiction, proves the claim. □

Theorem 8.7. *Let G be connected. Then $\chi(G) = \Delta(G) + 1$ if and only if either G is an odd cycle or a clique.*

Proof. ($\Longleftarrow$) Indeed, $\chi(C_{2k+1}) = 3$, $\Delta(C_{2k+1}) = 2$, and $\chi(K_n) = n$, $\Delta(K_n) = n - 1$.

($\Longrightarrow$) Assume that $k = \chi(G)$.

We may suppose that G is k-critical. Indeed, assume the claim holds for k-critical graphs. Let $k = \Delta(G) + 1$, and let $H \subset G$ be a k-critical proper subgraph. Since $\chi(H) = k = \Delta(G)+1 > \Delta(H)$, we must have $\chi(H) = \Delta(H)+1$, and thus H is a clique or an odd cycle. Now G is connected, and therefore there exists an edge $uv \in E_G$ with $u \in V_H$ and $v \notin V_H$. But then $d_G(u) > d_H(u)$, and $\Delta(G) > \Delta(H)$, since $H = K_n$ or $H = C_n$.

Let then G be any k-critical graph for $k \geq 2$. By Lemma 8.5, it is 2-connected. If G is an *even* cycle, then $k = 2 = \Delta(G)$. Suppose now that G is neither a clique nor a cycle (odd or even). We show that $\chi(G) \leq \Delta(G)$.

By Lemma 8.6, there exist $v_1, v_2 \in V_G$ with $d_G(v_1, v_2) = 2$, say $v_1w, wv_2 \in E_G$ with $v_1v_2 \notin E_G$, such that $H = G - \{v_1, v_2\}$ is connected. Order $V_H = \{v_3, v_4, \dots, v_n\}$ such that $v_n = w$, and for all $i \geq 3$,

$$d_H(v_i, w) \geq d_H(v_{i+1}, w)\,.$$

Therefore for each $i \in [1, n-1]$, we find at least one $j > i$ such that $v_iv_j \in E_G$ (possibly $v_j = w$). In particular, for all $1 \leq i < n$,

$$|N_G(v_i) \cap \{v_1, \dots, v_{i-1}\}| < d_G(v_i) \leq \Delta(G)\,. \tag{8.1}$$

Then colour $v_1, v_2, \dots, v_n$ in this order as follows: $\alpha(v_1) = 1 = \alpha(v_2)$ and

$$\alpha(v_i) = \min\{r \mid r \neq \alpha(v_j) \text{ for all } v_j \in N_G(v_i) \text{ with } j < i\}\,.$$

The colouring α is proper.

By (8.1), $\alpha(v_i) \leq \Delta(G)$ for all $i \in [1, n-1]$. Also, $w = v_n$ has two neighbours, v_1 and v_2, of the same colour 1, and since v_n has at most $\Delta(G)$ neighbours, there is an available colour for v_n, and so $\alpha(v_n) \leq \Delta(G)$. This shows that G has a proper $\Delta(G)$-colouring, and, consequently, $\chi(G) \leq \Delta(G)$. □

Example 8.8. Suppose we have n objects $V = \{v_1, \dots, v_n\}$, some of which are not compatible (like chemicals that react with each other, or worse, graph theorists who will fight during a conference). In the **storage problem** we would like to find a partition of the set V with as few classes as possible such that no class contains two incompatible elements. In graph theoretical terminology we consider the graph $G = (V, E)$, where $v_iv_j \in E$ just in case v_i and v_j are incompatible, and we would like to colour the vertices of G properly using as few colours as possible. This problem requires that we find $\chi(G)$.

Unfortunately, no good algorithms are known for determining $\chi(G)$, and, indeed, the chromatic number problem is NP-complete. Already the problem if $\chi(G) = 3$ is NP-complete. (However, as we have seen, the problem whether $\chi(G) = 2$ has a fast algorithm.) □

The chromatic polynomial

A given graph G has many different proper vertex colourings $\alpha\colon V_G \to [1,k]$ for sufficiently large natural numbers k. Indeed, see Lemma 8.2 to be certain on this point.

DEFINITION. The **chromatic polynomial** of G is the function $\chi_G\colon \mathbb{N} \to \mathbb{N}$, where

$$\chi_G(k) = |\{\alpha \mid \alpha\colon V_G \to [1,k] \text{ a proper colouring}\}| . \qquad \square$$

This notion was introduced by BIRKHOFF (1912), BIRKHOFF AND LEWIS (1946), to attack the famous 4-Colour Theorem, but its applications have turned out to be elsewhere.

If $k < \chi(G)$, then clearly $\chi_G(k) = 0$, and, indeed,

$$\chi(G) = \min\{k \mid \chi_G(k) \neq 0\} .$$

Therefore, if we can find the chromatic polynomial of G, then we easily compute the chromatic number $\chi(G)$ just by evaluating $\chi_G(k)$ for $k = 1, 2, \dots$ until we hit a nonzero value. Theorem 8.9 will give the tools for constructing χ_G.

As an example consider the clique K_4 on $\{v_1, v_2, v_3, v_4\}$. Let $k \geq \chi(K_4) = 4$. The vertex v_1 can be first given any of the k colours, after which $k-1$ colours are available for v_2. Then v_3 has $k-2$ and finally v_4 has $k-3$ available colours. Therefore there are $k(k-1)(k-2)(k-3)$ different ways to properly colour K_4 with k colours, and so

$$\chi_{K_4}(k) = k(k-1)(k-2)(k-3) .$$

On the other hand, in the discrete graph $\overline{K}_4$ has no edges, and thus any k-colouring is a proper colouring. Therefore

$$\chi_{\overline{K}_4}(k) = k^4 .$$

At this point we warn that the considered method for checking the number of possibilities to colour a 'next vertex' is exceptional, and for more nonregular graphs it should be avoided.

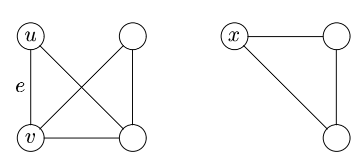
DEFINITION. Let G be a graph, $e = uv \in E_G$, and let $x = x(uv)$ be a new **contracted vertex**. The graph $G * e$ on

$$V_{G*e} = (V_G - \{u, v\}) \cup \{x\}$$

is obtained from G by **contracting** the edge e, when

$$E_{G*e} = \{f \mid f \in E_G,\ f \text{ has no end } u \text{ or } v\} \cup \{wx \mid wu \in E_G \text{ or } wv \in E_G\} . \qquad \square$$

Hence $G * e$ is obtained by introducing a new vertex x, and by replacing all edges wu and wv by wx, and the vertices u and v are deleted. (Of course, no loops or parallel edges are allowed in the new graph $G * e$.)



Theorem 8.9. *Let G be a graph, and let $e \in E_G$. Then*

$$\chi_G(k) = \chi_{G-e}(k) - \chi_{G*e}(k)$$

Proof. Let $e = uv$. The proper k-colourings $\alpha\colon V_G \to [1, k]$ of $G{-}e$ can be divided into two disjoint cases, which together show that $\chi_{G-e}(k) = \chi_G(k) + \chi_{G*e}(k)$:

(1) If $\alpha(u) \neq \alpha(v)$, then α corresponds to a unique proper k-colouring of G, namely α. Hence the number of such colourings is $\chi_G(k)$.

(2) If $\alpha(u) = \alpha(v)$, then α corresponds to a unique proper k-colouring of $G * e$, namely α, when we set $\alpha(x) = \alpha(u)$ for the contracted vertex $x = x(uv)$. Hence the number of such colourings is $\chi_{G*e}(k)$. □

Theorem 8.10. *The chromatic polynomial is a polynomial.*

Proof. The proof is by induction on ε_G. Indeed, $\chi_{\overline{K}_n}(k) = k^n$ for the discrete graph, and for two polynomials P_1 and P_2, also $P_1 - P_2$ is a polynomial. The claim follows from Theorem 8.9, since there $G{-}e$ and $G * e$ have less edges than G. □

The components of a graph can be coloured independently, and so

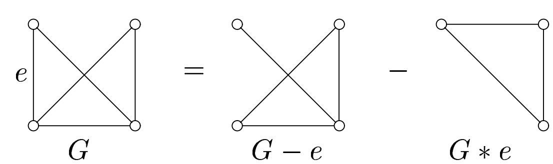
Lemma 8.11. *Let the graph G have the components $G_1, G_2, \dots, G_m$. Then*

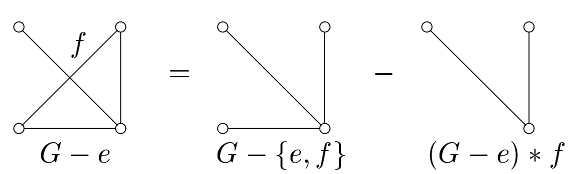
$$\chi_G(k) = \chi_{G_1}(k)\chi_{G_2}(k) \dots \chi_{G_m}(k)\,.$$

Theorem 8.12. *Let T be a tree of order n. Then $\chi_T(k) = k(k-1)^{n-1}$.*

Proof. We use induction on n. For $n \le 2$, the claim is obvious. Suppose that $n \ge 3$, and let $e = vu \in E_T$, where v is a leaf. By Theorem 8.9, $\chi_T(k) = \chi_{T-e}(k) - \chi_{T*e}(k)$. Here $T * e$ is a tree of order $n-1$, and thus, by the induction hypothesis, $\chi_{T*e}(k) = k(k-1)^{n-2}$. The graph $T{-}e$ consists of the isolated v and a tree of order $n-1$. By Lemma 8.11, and the induction hypothesis, $\chi_{T-e}(k) = k \cdot k(k-1)^{n-2}$. Therefore $\chi_T(k) = k(k-1)^{n-1}$. □

As an example consider the graph G of order 4 from the above.





Theorem 8.9 reduces the computation of χ_G to the discrete graphs. However, we know the chromatic polynomials for trees (and cliques, as an exercise), and so there is no need to prolong the reductions beyond these. In our example, we have obtained

$$\chi_{G-e}(k) = \chi_{G-\{e,f\}}(k) - \chi_{(G-e)*f}(k) = k(k-1)^3 - k(k-1)^2 = k(k-1)^2(k-2)\,,$$

and so

$$\begin{aligned}\chi_G(k) &= \chi_{G-e}(k) - \chi_{G*e}(k) = k(k-1)^2(k-2) - k(k-1)(k-2)\\ &= k(k-1)(k-2)^2 = k^4 - 5k^3 + 8k^2 - 4k\,.\end{aligned}$$

For instance, for 3 colours, there are 6 proper colourings of the given graph.

Research problem: The chromatic polynomials pose many difficult problems. It is difficult to determine χ_G of a given graph, since the reduction method is time consuming. Also, there is known no characterization, which would tell from any polynomial $P(k)$ whether it is a chromatic polynomial of some graph. For instance, the polynomial $k^4 - 3k^3 + 3k^2$ is not a chromatic polynomial of any graph, but it seems to satisfy the general properties (that are known or conjectured) of these polynomials. Reed (1968) conjectured that the coefficients of a chromatic polynomial should first increase and then decrease in absolute value. Reed (1968) and Tutte (1974) proved that for each G of order $\nu_G = n$:

- The degree of $\chi_G(k)$ equals n.
- The coefficient of k^n equals 1.
- The coefficient of k^{n-1} equals $-\varepsilon_G$.
- The constant term is 0.
- The coefficients alternate in sign.
- $\chi_G(m) \le m(m-1)^n - 1$ for all positive integers m, when G is connected.
- $\chi_G(x) \neq 0$ for all real numbers $0 < x < 1$.

Perfect graphs*

Denote by

$$\omega(G) = \max\{m \mid K_m \subseteq G\}$$

the size of a **maximal clique** of G.

It is clear that $\chi(G) \ge \omega(G)$ for all graphs. There can be a strict inequality here. Indeed, for the odd cycles, we have $\chi(C_{2k+1}) = 3 > 2 = \omega(C_{2k+1})$ for all $k \ge 2$, and for the complements of these odd cycles, we have $\chi(\overline{C}_{2k+1}) = k+1 > k = \omega(\overline{C}_{2k+1})$.

Definition. A **Berge graph** is a graph that does not contain an odd cycle C_{2k+1} nor the complement $\overline{C}_{2k+1}$ as an induced subgraph for any $k \ge 2$. A graph G is said to be **perfect**, if $\chi(H) = \omega(H)$ for all induced subgraphs $H = G[A]$ of G. □

The Berge graphs were originally motivated by Shannon's work on optimal codes.

Research problem. It was conjectured by BERGE (1962) that *his graphs are precisely the perfect graphs.* This popular conjecture is known as the **perfect graph conjecture**.

LOVÁSZ proved in 1972 the following **Perfect Graph Theorem**:

Theorem 8.13. *A graph G is perfect if and only if its complement $\overline{G}$ is perfect.*

In fact, LOVÁSZ proved a stronger result:

Theorem 8.14. *A graph G is perfect if and only if*

$$\omega(G[A]) \cdot \alpha(G[A]) \geq |A|$$

for all $A \subseteq V_G$, where

$$\alpha(H) = \max\{m \mid \overline{K}_m \subseteq G\}$$

is the size of a largest stable subset of H.

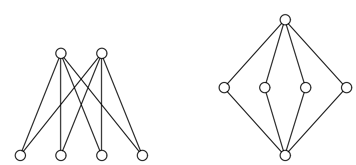
9 Planar Graphs

The plane representations of graphs are by no means unique. Indeed, a graph G can be drawn in arbitrarily many different ways. Also, the properties of a graph are not necessarily immediate from one representation, but may be apparent from another. There are, however, important families of graphs, the **surface graphs**, that rely on the (topological or geometrical) properties of the drawings of graphs. We restrict ourselves in this chapter to the most natural of these, the planar graphs. The geometry of the plane will be treated intuitively.

A planar graph will be a graph that can be drawn in the plane so that no two edges intersect with each other. Such graphs are used, *e.g.*, in the design of electrical (or similar) circuits, where one tries to (or has to) avoid crossing the wires or laser beams. Planar graphs come into use also in some parts of mathematics, especially in group theory and topology.

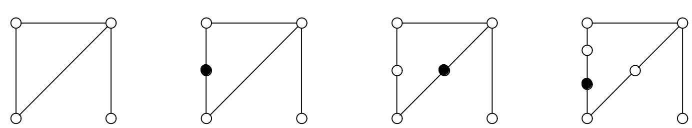
Definition

Definition. A graph G is a **planar graph**, if it has a plane figure $P(G)$, called the **plane embedding** of G, where the lines (or continuous curves) corresponding to the edges do not intersect each other except at their ends. □



The complete bipartite graph $K_{2,4}$ is a planar graph.

Definition. An edge $e = uv \in E_G$ is **subdivided**, when it is replaced by a path $u \longrightarrow x \longrightarrow v$ of length two by introducing a *new* vertex x. A **subdivision** H of a graph G is obtained from G by a sequence of subdivisions. □



The following result is clear.

Lemma 9.1. *A graph is planar if and only if its subdivisions are planar.*

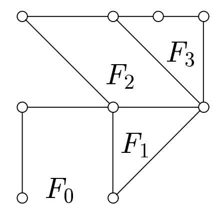
Geometric properties

It is clear that the graph theoretical properties of G are inherited by all of its plane embeddings. For instance, the way we draw a graph G in the plane does not change its maximum degree or its chromatic number. More importantly, there are – as we shall see – some nontrivial topological (or geometric) properties that are shared by the plane embeddings.

We recall first some elements of the plane geometry. Let F be an open set of the plane $\mathbb{R} \times \mathbb{R}$. Then F is a **region**, if any two points $x, y \in F$ can be joined by a continuous curve the points of which are all in F. The **boundary** $\partial(F)$ of a region F consists of those points for which every neighbourhood contains points from F and its complement.

Let G be a planar graph, and $P(G)$ one of its plane embeddings. Regard now each edge $e = uv \in E_G$ as a line from u to v. The set $\mathbb{R} \times \mathbb{R} - E_G$ is open, and it is divided into a finite number of disjoint regions, called the **faces** of $P(G)$.

DEFINITION. A face of $P(G)$ is an **interior face**, if it is bounded. The (unique) face that is unbounded is called the **exterior face** of $P(G)$. The edges that surround a face F constitute the boundary $\partial(F)$ of F. The **exterior boundary** is the boundary of the exterior face. □



Embeddings $P(G)$ satisfy some properties that we accepts at face value.

Lemma 9.2. *Let $P(G)$ be a plane embedding of a planar graph G.*

(i) *Two different faces F_1 and F_2 are disjoint, and their boundaries can intersect only on edges.*
(ii) *$P(G)$ has a unique exterior face.*
(iii) *Each edge e belongs to the boundary of at most two faces.*
(iv) *Each cycle of G surrounds (that is, its interior contains) at least one internal face of $P(G)$.*
(v) *A bridge e of G belongs to the boundary of only one face.*
(vi) *An edge e that is not a bridge belongs to the boundary of exactly two faces.*

If $P(G)$ is a plane embedding of a graph G, then so is any drawing $P'(G)$ which is obtained from $P(G)$ by an injective mapping of the plane that preserves continuous curves. This means, in particular, that *every planar graph has a plane embedding inside any geometric circle of arbitrarily small radius, or inside any geometric triangle.*

Euler's formula

Lemma 9.3. *A plane embedding $P(G)$ of a planar graph G has no interior faces if and only if G is acyclic, that is, if and only if the components of G are trees.*

Proof. This is clear from Lemma 9.2. □

The next **Euler's formula** was proved by LEGENDRE (1794).

Theorem 9.4. *Let G be a connected planar graph, and let $P(G)$ be any of its plane embeddings. Then*

$$\nu_G - \varepsilon_G + \varphi = 2\,,$$

where φ is the number of faces of $P(G)$. In particular, every plane embedding of G has the same number $\varphi = \varepsilon_G - \nu_G + 2$ of faces.

Proof. We show the claim by induction on φ. Since each $P(G)$ has an exterior face, $\varphi \geq 1$. If $\varphi = 1$, then, by Lemma 9.3, there are no cycles in G, and since G is connected, it is a tree. In this case we know that $\varepsilon_G = \nu_G - 1$, and the claim holds.

Suppose then that the claim is true for all plane embeddings with less than φ faces for $\varphi \geq 2$. Let $P(G)$ be a plane embedding of a connected planar graph such that $P(G)$ has φ faces.

Let $e \in E_G$ be an edge that is not a bridge. The subgraph $G-e$ is planar with a plane embedding $P(G-e) = P(G)-e$ obtained by simply erasing the edge e. Now $P(G-e)$ has $\varphi-1$ faces, since the two faces of $P(G)$ that are separated by e are merged into one face of $P(G-e)$. By the induction hypothesis, $\nu_{G-e} - \varepsilon_{G-e} + (\varphi - 1) = 2$, and hence $\nu_G - (\varepsilon_G - 1) + (\varphi - 1) = 2$, and the claim follows. □

Kuratowski's theorem

Theorem 9.8 will give a simple criterion for planarity of graphs.

Lemma 9.5. *If G is a planar graph of order $\nu_G \geq 3$, then $\varepsilon_G \leq 3\nu_G - 6$. Moreover, if G has no triangles C_3, then $\varepsilon_G \leq 2\nu_G - 4$.*

Proof. If G is disconnected with components G_i, for $i \in [1, k]$, and if the claim holds for these smaller (necessarily planar) graphs G_i, then it holds for G, since

$$\varepsilon_G = |E_G| = \sum_{i=1}^{\nu_G} |E_{G_i}| \leq 3\sum_{i=1}^{\nu_G} |V_{G_i}| - 6k = 3\nu_G - 6k \leq 3\nu_G - 6\,.$$

It is thus sufficient to prove the claim for connected planar graphs.

Also, the case where $\varepsilon_G \leq 2$ is clear. Suppose thus that $\varepsilon_G \geq 3$.

Each face F of an embedding $P(G)$ contains at least three edges on its boundary $\partial(F)$. Hence $3\varphi \leq 2\varepsilon_G$, since each edge lies on at most two faces. The first claim follows from Euler's formula.

The second claim is proved similarly except that, in this case, each face F of $P(G)$ contains at least four edges on its boundary (when G is connected and $\varepsilon_G \geq 4$). □

Theorem 9.6. *If G is a planar graph, then $\delta(G) \leq 5$.*

Proof. If $\nu_G \leq 2$, then there is nothing to prove. Suppose $\nu_G \geq 3$. By the handshaking lemma and the previous lemma,

$$\delta(G) \cdot \nu_G \leq \sum_{v \in V_G} d_G(v) = 2\varepsilon_G \leq 6\nu_G - 12\,.$$

It follows that $\delta(G) \leq 5$. □

Corollary 9.7. *K_5 and $K_{3,3}$ are not planar graphs.*

Proof. By Lemma 9.5, a planar graph of order 5 has at most 9 edges, but K_5 has 5 vertices and 10 edges. By the second claim of Lemma 9.5, a triangle-free planar graph of order 6 has at most 8 edges, but $K_{3,3}$ has 6 vertices and 9 edges. □

The graphs K_5 and $K_{3,3}$ are the smallest nonplanar graphs, and, by Lemma 9.1, if G contains a subdivision of K_5 or $K_{3,3}$ as a subgraph, then G is not planar. A remarkable theorem due to Kuratowski (1930) states that also the converse is true.

Theorem 9.8. *A graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ as a subgraph.*

We prove this result along the lines of Thomassen (1981) using 3-connectivity.

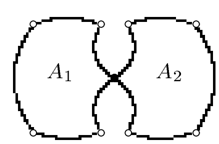
Definition. A graph G is a **Kuratowski graph**, if it is a subdivision of K_5 or $K_{3,3}$. □

Lemma 9.9. *Let $E \subseteq E_G$ be the set of the boundary edges of a face F in a plane embedding of G. Then there exists a plane embedding $P(G)$, where the edges of E are on the exterior boundary.*

Proof. This is a geometric proof. Choose a circle that contains every point of the plane embedding (including all points of the edges) such that the centre of the circle is inside the given face. Then use geometric inversion with respect to this circle. This will map the given face as the exterior face of the image plane embedding. □

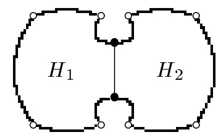
Lemma 9.10. *Let G be a nonplanar graph without Kuratowski graphs such that ε_G is minimal in this respect. Then G is 3-connected.*

Proof. We show first that G is 2-connected. On the contrary, assume that v is a cut vertex of G, and let $A_1, \dots, A_k$ be the components of $G-v$.
Since G is minimal nonplanar with respect to ε_G, the subgraphs $G_i = G[A_i \cup \{v\}]$ have plane embeddings $P(G_i)$, where the vertex v is on the exterior boundary. We can glue these plane embeddings together at v to obtain a plane embedding of G, and this will contradict the choice of G.



Assume then that G has a separating set $S = \{u, v\}$. Let G_1 and G_2 be any subgraphs of G such that $E_G = E_{G_1} \cup E_{G_2}$, $S = V_{G_1} \cap V_{G_2}$, and both G_1 and G_2 contain a component of $G{-}S$. Since G is 2-connected (by the above), there are paths $u \xrightarrow{\star} v$ in G_1 and G_2. Indeed, both u and v are adjacent to a vertex of each component of $G{-}S$. Let $H_i = G_i + uv$. (Maybe $uv \in E_G$.)

If both H_1 and H_2 are planar, then, by Lemma 9.9, they have plane embeddings, where uv is on the exterior boundary. It is now easy to glue H_1 and H_2 together on the edge uv to obtain a plane embedding of $G + uv$, and thus of G.

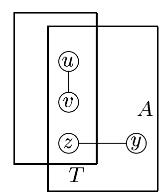


We conclude that H_1 or H_2 is nonplanar, say H_1. Now $\varepsilon_{H_1} < \varepsilon_G$, and so, by the minimality of G, H_1 contains a Kuratowski graph H. However, there is a path $u \xrightarrow{\star} v$ in H_2, since $G_2 \subseteq H_2$. This path can be regarded as a subdivision of uv, and thus G contains a Kuratowski graph. This contradiction shows that G is 3-connected. □

Lemma 9.11. *Let G be a 3-connected graph of order $\nu_G \geq 5$. Then there exists an edge $e \in E_G$ such that the contraction $G * e$ is 3-connected.*

Proof. On the contrary suppose that for any $e \in E_G$, the graph $G * e$ has a separating set S with $|S| = 2$. Let $e = uv$, and let $x = x(uv)$ be the contracted vertex. Necessarily $x \in S$, say $S = \{x, z\}$ (for, otherwise, S would separate G already). Therefore $T = \{u, v, z\}$ separates G. Assume that e and S are chosen such that $G{-}T$ has a component A with the least possible number of vertices.

There exists a vertex $y \in A$ with $zy \in E_G$. (Otherwise $\{u, v\}$ would separate G.) The graph $G * (zy)$ is not 3-connected by assumption, and hence, as in the above, there exists a vertex w such that $R = \{z, y, w\}$ separates G. We do not excluded the case, where $w = u$ or $w = v$.



Since $uv \in E_G$, $G{-}R$ has a component B such that $u, v \notin B$. There exists a vertex $y' \in B$ such that $yy' \in E_G$. (Again, otherwise $\{z, w\}$ would separate G.) However, $(G{-}T)[B]$ is connected, and so $B \subseteq A$. The inclusion is proper, since $y \notin B$. Hence $|B| < |A|$, and this contradicts the choice of A. □

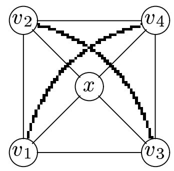
By the next lemma, a Kuratowski graph cannot be created by contractions.

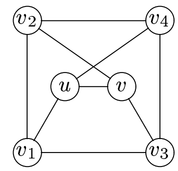
Lemma 9.12. *Let G be a graph. If for some $e \in E_G$ the contraction $G * e$ has a Kuratowski subgraph, then so does G.*

Proof. The proof consists of several cases depending on the Kuratowski graph, and how the subdivision is made. We do not consider the details of these cases.

Let H be a Kuratowski graph of $G * e$, where $x = x(uv)$ is the contracted vertex for $e = uv$. If $d_H(x) = 2$, then the claim is obviously true. Suppose then that $d_H(x) = 3$ or 4. If there exists at most one edge $xy \in E_H$ such that $uy \in E_G$ (or $vy \in E_G$), then one easily sees that G contains a Kuratowski graph.

There remains only one case, where H is a subdivision of K_5, and both u and v have 3 neighbours in the subgraph of G corresponding to H. In this case, G contains a subdivision of $K_{3,3}$. □





Lemma 9.13. *Every 3-connected graph G without Kuratowski subgraphs is planar.*

Proof. The proof is by induction on ν_G. The only 3-connected graph of order 4 is the planar graph K_4. Therefore we can assume that $\nu_G \geq 5$.

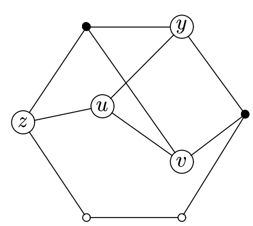
By Lemma 9.11, there exists an edge $e = uv \in E_G$ such that $G{*}e$ (with a contracted vertex x) is 3-connected. By Lemma 9.12, $G * e$ has no Kuratowski subgraphs, and hence $G * e$ has a plane embedding $P(G * e)$ by the induction hypothesis. Consider the part $P(G * e){-}x$, and let C be the boundary of the face of $P(G * e){-}x$ containing x (in $P(G * e)$). Here C is a cycle of G (since G is 3-connected).

Now since $G{-}\{u, v\} = (G * e){-}x$, $P(G * e){-}x$ is a plane embedding of $G{-}\{u, v\}$, and all the neighbours of u and v in G belong to the cycle C. Let the neighbours of v be $v_1, v_2, \ldots, v_k$ in order along the cycle C. The path along C from v_i to v_{i+1} (indices modulo k) is referred to as a *segment*. We obtain a plane embedding of $G{-}u$ by drawing (straight) edges vv_i for $1 \leq i \leq k$.

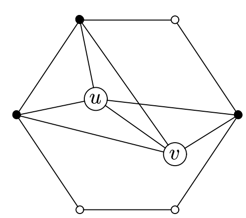
There are four cases to consider:

(1) If the neighbours of u (except v) belong to the same segment (v_i, v_{i+1}) of C, then, clearly, G has a plane embedding (obtained from $P(G){-}u$ by putting u inside the triangle (v, v_i, v_{i+1}) and by drawing the edges with an end u inside this triangle).

(2) If u has a neighbour $y \neq v$ such that $y \notin N_G(v)$, then y is in a segment (v_i, v_{i+1}) for some i, and $y \notin \{v_i, v_{i+1}\}$. By Case (1), there exists a $z \in N_G(u)$ with $z \neq v$ such that z is not in the segment (v_i, v_{i+1}). Now, $\{u, v_i, v_{i+1}\} \cup \{v, z, y\}$ form a subdivision of $K_{3,3}$.

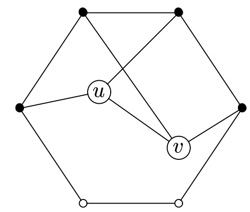


We can assume then that $N_G(u) - \{v\} \subseteq N_G(v)$.



(3) If u and v have (at least) three common neighbours v_i, v_j, v_m, then u, v, v_i, v_j, v_m give a subdivision of K_5.

(4) If u and v have exactly two common neighbours v_i and v_j for $i < j$, then, by Case (1), $j \neq i+1$ and $j+1 \neq i$ (modulo k). This means that $\{u, v_{i+1}, v_{j+1}\} \cup \{v, v_i, , v_j\}$ gives a subdivision $K_{3,3}$ in G.



□

Proof of Theorem 9.8. By Corollary 9.7 and Lemma 9.1, we need to show that each nonplanar graph G contains a Kuratowski subgraph. On the contrary, suppose that G is nonplanar with minimal ε_G such that G does not contain a Kuratowski subgraph. Then, by Lemma 9.10, G is 3-connected, and by Lemma 9.13, it is planar. This contradiction proves the claim. □

Maximal planar graphs

DEFINITION. A planar graph G is **maximal**, if $G+e$ is nonplanar for every $e \notin E_G$. □

Clearly, if we remove one edge from K_5, the result is a maximal planar graph. (However, if an edge is removed from $K_{3,3}$, the result is not maximal!)

Lemma 9.14. *Let F be a face of a plane embedding $P(G)$ that has at least four edges on its boundary. Then there are two nonadjacent vertices on the boundary of F.*

Proof. Assume that the set of the boundary vertices of F induces a clique K. The edges of K are on the boundary of or exterior to F (since F is a face.) Add a new vertex x inside F, and connect the vertices of K to x. The result is a plane embedding of a graph H with $V_H = V_G \cup \{x\}$ (that has G as its induced subgraph). The induced subgraph $H[K \cup \{x\}]$ is a clique, and since H is planar, we have $|K| < 4$ as required. □

Corollary 9.15. *If G is a maximal planar graph with $\nu_G \geq 3$, then G is triangulated, that is, every face of a plane embedding $P(G)$ has a boundary of exactly three edges.*

Theorem 9.16. *For a maximal planar graph G of order $\nu_G \geq 3$, $\varepsilon_G = 3\nu_G - 6$.*

Proof. Each face F of an embedding $P(G)$ is a triangle having three edges on its boundary. Hence $3\varphi = 2\varepsilon_G$, since there are now no bridges. The claim follows from Euler's formula. □

The chromatic number

We prove HEAWOOD's result (1890) that each planar graph is properly 5-colourable.

Lemma 9.17. *If G is a planar graph, then $\chi(G) \leq 6$.*

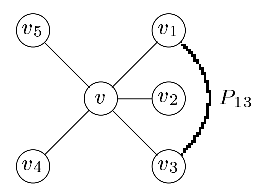
Proof. The proof is by induction on ν_G. Clearly, the claim holds for $\nu_G \leq 6$. By Theorem 9.6, a planar graph G has a vertex v with $d_G(v) \leq 5$. By the induction hypothesis, $\chi(G-v) \leq 6$. Since $d_G(v) \leq 5$, there is a colour i available for v in the 6-colouring of $G-v$, and so $\chi(G) \leq 6$. □

The proof of Heawood's theorem is partly geometric in nature.

Theorem 9.18. *If G is a planar graph, then $\chi(G) \leq 5$.*

Proof. Suppose the claim does not hold, and let G be a 6-critical planar graph. Recall that for k-critical graphs H, $\delta(H) \geq k-1$, and thus there exists a vertex v with $d_G(v) = \delta(G) \geq 5$. By Theorem 9.6, $d_G(v) = 5$.

Let α be a proper 5-colouring of the subgraph $G-v$. Such a colouring exists, because G is 6-critical. By assumption, $\chi(G) > 5$, and therefore for each $i \in [1,5]$, there exists a neighbour $v_i \in N_G(v)$ such that $\alpha(v_i) = i$. Suppose these neighbours v_i of v occur in the plane in the geometric order of the figure.



Consider the subgraph $G[i,j]$ made of colours i and j. The vertices v_i and v_j are in the same component of $G[i,j]$ (for, otherwise we interchange the colours i and j in the component containing v_j to obtain a recolouring of G, where v_i and v_j have the same colour i, and then recolour v with the remaining colour j).

Let $P_{ij} \colon v_i \xrightarrow{\star} v_j$ be a path in $G[i,j]$, and let $C = (vv_1)P_{13}(v_3v)$. By the geometric assumption, exactly one of v_2, v_4 lies inside the region enclosed by the cycle C. Now, the path P_{24} must meet C at some vertex of C, since G is planar. This is a contradiction, since the vertices of P_{24} are coloured by 2 and 4, but C contains no such colours. □

CUTHRIE conjectured in 1850s that *four colours suffice* for planar graphs. This conjecture resisted until APPEL AND HAKEN solved it in 1976. The disputed proof used 1200 hours of computer's time. A simplified proof by ROBERTSON, SANDERS, SEYMOUR AND THOMAS (1997) uses the computer much less than the original proof.

Theorem 9.19 (4-Colour Theorem). *If G is a planar graph, then $\chi(G) \leq 4$.*

Example 9.20. The 4-Colour Theorem has an interesting equivalent formulation in terms of combinatorics of words due to DESCARTES AND DESCARTES (1968). A word (sequence) $x_0x_1 \dots x_n$ of letters a, b, c, d is said to be **Cartesian**, if no consecutive letters are the same and the even numbered letters $x_0x_2x_4\dots$ form a Cartesian word. The equivalent result is now: *for all n and $0 \leq i_0 < i_1 < \dots < i_m \leq n$, there exists a Cartesian word $x_0x_1 \dots x_n$ such that also $x_{i_0}x_{i_1}\dots x_{i_m}$ is Cartesian.* □

List colourings*

DEFINITION. Let G be a graph so that each of its vertices v is given a list (set) $\Lambda(v)$ of colours. A proper colouring $\alpha\colon V_G \to [1,m]$ of G is a (Λ-)**list colouring**, if each vertex v gets a colour from its list, $\alpha(v) \in \Lambda(v)$.

The **list chromatic number** $\chi_\ell(G)$ is the smallest integer k such that G has a Λ-list colouring for all lists of size k, $|\Lambda(v)| = k\}$.
Also, G is k-**choosable**, if $\chi_\ell(G) \le k$. □

We have $\chi(G) \le \chi_\ell(G)$, but equality does not hold in general. However, it was proved by VIZING (1976) and ERDÖS, RUBIN AND TAYLOR (1979) that

$$\chi_\ell(G) \le \Delta(G) + 1 .$$

For planar graphs we *do not* have a '4-list colour theorem'. Indeed, it was shown by VOIGT (1993) that there exists a planar graph with $\chi_\ell(G) = 5$. At the moment, the smallest such a graph was produced by MIRZAKHANI (1996), and it is of order 63. However, THOMASSEN (1994) proved:

Theorem 9.21. *Let G be a planar graph. Let the exterior vertices have any lists of 3 colours and the other vertices any lists of 5 colours. Then G can be list coloured using these lists. In particular, $\chi_\ell(G) \le 5$.*

Colourings of maps*

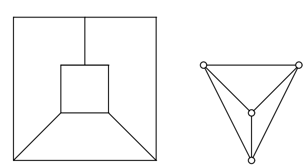
The **map-colouring problem** is stated as follows. There are several countries with common borders. We wish to colour each country so that no neighbouring countries get the same colour. *How many colours are needed?* (A border between two countries is assumed to have a positive length – in particular, countries that have only one point in common are not really neighbours).

Formally, we define a **map** as a connected planar (embedding of a) graph with no bridges. The edges of this graph represent the boundaries between countries. Hence a country is a face of the map, and two neighbouring countries share a common edge (not just a single vertex). We deny bridges, because a bridge in a such a map would be a boundary inside a country.

The map-colouring problem is restated as follows:

How many colours are needed for the faces of a plane embedding so that no adjacent faces obtain the same colour.

The illustrated map is 4-coloured, and no fewer colours are possible for this map, because every two faces have a common border.



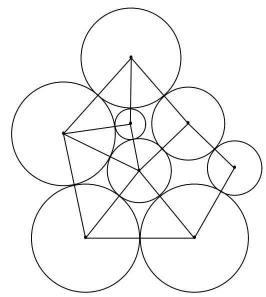
Using a 'dual graph' of a map we can state the map-colouring problem in new form: *What is the chromatic number of a planar graph?* By the 4-Colour Theorem it is at most four.

Kissing circles*

We state an interesting result of WAGNER (1936), the proof of which can be deduced from the above proof of Kuratowski's theorem.

Theorem 9.22. *A planar graph G has a plane embedding, where the edges are straight lines.*

We say that two circles **kiss** in the plane, if they intersect in one point and their interiors do not intersect. For a set of circles, we draw a graph by putting an edge between two midpoints of kissing circles.



The following improvement of the above theorem is due to KOEBE (1936), and it was rediscovered independently by ANDREEV (1970) and THURSTON (1985).

Theorem 9.23. *A graph is planar if and only if it is a kissing graph of circles.*

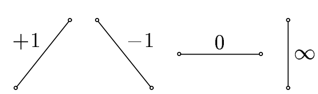
Graphs can be represented as plane figures in many different ways. For this, consider a set S of curves of the plane (that are continuous between their end points). The **string graph** of S is the graph $G = (S, E)$, where $uv \in E$ if and only if the curves u and v intersect. At first it might seem that every graph is a string graph, but this is not the case.

It is known that all planar graphs are string graphs (this is a trivial result).

Research Problem. A graph is a **line segment graph** if it is a string graph for a set L of straight line segments in the plane. *Is every planar graph a line segment graph for some set L of lines?*

Note that there are also nonplanar graphs that are line segment graphs. Indeed, all cliques are such graphs.

The above question remains open even in the case when the slopes of the lines are $+1$, -1, 0 and ∞:

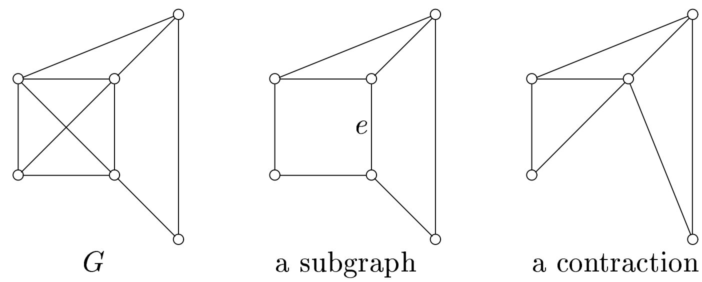


A positive answer to this 4-slope problem for planar graphs would give a proof of the 4-Colour Theorem.

The Minor Theorem*

DEFINITION. A graph H is a **minor** of G, denoted by $H \preccurlyeq G$, if H is isomorphic to a graph obtained from a *subgraph* of G by successively contracting edges. □

A recent result of ROBERTSON AND SEYMOUR (1983-2000) on graph minors is (one of) the deepest results in the history of graph theory. The proof, or a detailed introduction to the topic, goes much beyond these (at the moment, any) lecture notes. Indeed, the proof of Theorem 9.24 is around 500 pages long.



Note that every subgraph $H \subseteq G$ is a minor, $H \preccurlyeq G$.

The following properties of the minor relation are easily established:

(i) $G \preccurlyeq G$,
(ii) $H \preccurlyeq G$ and $G \preccurlyeq H$ imply $G \cong H$,
(iii) $H \preccurlyeq L$ and $L \preccurlyeq G$ imply $H \preccurlyeq G$.

The conditions (i) and (iii) ensure that the relation $\preccurlyeq$ is a **quasi-order**, that is, it is reflexive and transitive. It turns out to be a **well-quasi-order**, that is, every infinite sequence $G_1, G_2, \dots$ of graphs has two graphs G_i and G_j with $i < j$ such that $G_i \preccurlyeq G_j$.

Theorem 9.24 (Minor Theorem). *The minor order $\preccurlyeq$ is a well-quasi-order on graphs. In particular, in any infinite family $\mathcal{F}$ of graphs, one of the graphs is a (proper) minor of another.*

Each property $\mathcal{P}$ of graphs defines a family of graphs, namely, the family of those graphs that satisfy this property.

DEFINITION. A family $\mathcal{F}$ of graphs is said to be **minor closed**, if every minor H of a graph $G \in \mathcal{F}$ is also in $\mathcal{F}$. A property $\mathcal{P}$ of graphs is said to be **inherited by minors**, if all minors of a graph G satisfy $\mathcal{P}$ whenever G does. □

The following lemma is obvious from the previous considerations.

Lemma 9.25. *A family of graphs is minor closed if and only if it closed under taking subgraphs and contractions.*

Example 9.26. The following families of graphs are minor closed: the family of

(i) all graphs,
(ii) planar graphs (and their generalizations to other surfaces),
(iii) acyclic graphs.

The acyclic graphs include all trees. However, the family of trees is not closed under taking subgraphs, and thus it is not minor closed. More importantly, the subgraph order of trees ($T_1 \subseteq T_2$) is *not* a well-quasi-order. □

WAGNER (1937) proved a minor version of Kuratowski's theorem:

Theorem 9.27. *A graph G is nonplanar if and only if $K_5 \preccurlyeq G$ or $K_{3,3} \preccurlyeq G$.*

ROBERTSON AND SEYMOUR (1998) proved the **Wagner's conjecture**:

Theorem 9.28 (Minor Theorem 2)**.** *Let $\mathcal{P}$ be a property of graphs inherited by minors. Then there exists a* finite set *$\mathcal{F}$ of graphs such that G satisfies $\mathcal{P}$ if and only if G does not have a minor from $\mathcal{F}$.*

One of the impressive application of Theorem 9.28 concerns embeddings of graphs on surfaces, see the next chapters. By Theorem 9.28, one can test (with a *fast* algorithm) whether a graph can be embedded onto a surface.

Every graph can be drawn in the 3-dimensional space without crossing edges. An old problem asks if there exists an algorithm that would determine whether a graph can be drawn so that its cycles do not form (nontrivial) knots. This problem is solved by the above results, since the property 'knotless' is inherited by minors: there *exists* a fast algorithm to do the job. However, this algorithm is not known!

Research Problem. HADWIGER conjectured in 1943 that for every graph G,

$$K_{\chi(G)} \preccurlyeq G\,,$$

that is, *if $\chi(G) \geq r$, then G has a clique K_r as its minor.* The conjecture is trivial for $r = 2$, and it is known to hold for all $r \leq 6$. The cases for $r = 5$ and 6 follow from the 4-Colour Theorem.

10 Genus of a Graph*

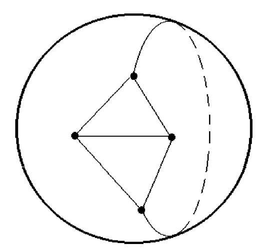
A graph is planar, if it can be drawn in the plane without crossing edges. A plane is an important special case of a surface. In this section we study shortly (and mostly intuitively) drawing graphs on other surfaces.

There are quite many interesting surfaces many of which are rather difficult to draw. We shall study only the 'easy surfaces' – the orientable surfaces. There are also non-orientable surfaces (like the Klein bottle).

Sphere

DEFINITION. In general, if S is a surface, then a graph G has an S-**embedding**, if G can be drawn on S without crossing edges. □

Let S_0 be (the surface of) a **sphere**. According to the next theorem a sphere has exactly the same embeddings as do the plane. In the one direction the claim is obvious: if G is a planar graph, then it can be drawn on a bounded area of the plane (without crossing edges), and this bounded area can be ironed on the surface of a large enough sphere.



Clearly, if a graph can be embedded on one sphere, then it can be embedded on any sphere – the size of the sphere is of no importance. On the other hand, if G is embeddable on a sphere S_0, then there is a small area of the sphere, where there are no points of the edges. We then puncture the sphere at this area, and stretch it open until it looks like a region of the plane. In this process no crossings of edges can be created, and hence G is planar.

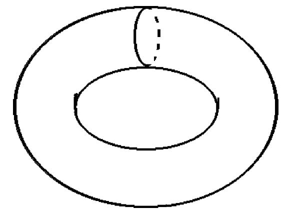
Theorem 10.1. *A graph G has an S_0-embedding if and only if it is planar.*

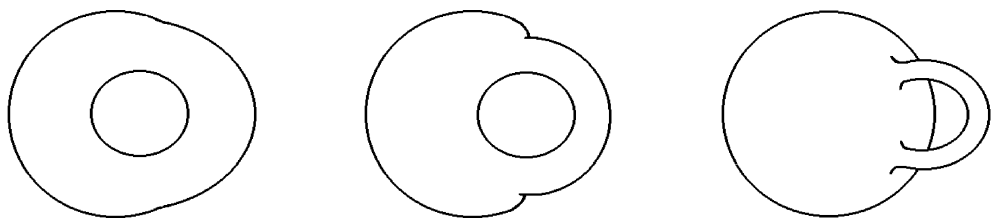
Therefore instead of planar embeddings we can equally well study embeddings of graphs on a sphere. This is sometimes convenient, since the sphere is closed and it has no boundaries. Most importantly, a planar graph drawn on a sphere has no exterior face – all faces are bounded (by edges).

If a sphere is deformed by pressing or stretching, its embeddability properties will remain the same. In topological terms the surface has been distorted by a continuous transformation.

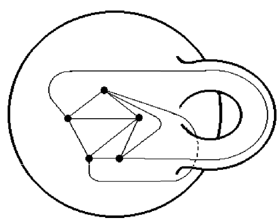
Torus

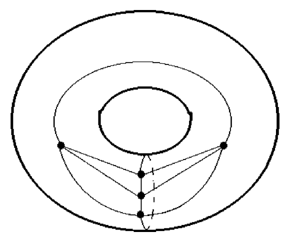
Consider next a surface which is obtained from the sphere S_0 by pressing a hole in it. This is a **torus** S_1 (or an **orientable surface of genus** 1). The S_1-embeddable graphs are said to have **genus** equal to 1.
Sometimes it is easier to consider handles than holes: a torus S_1 can be deformed (by a continuous transformation) into a **sphere with a handle**.



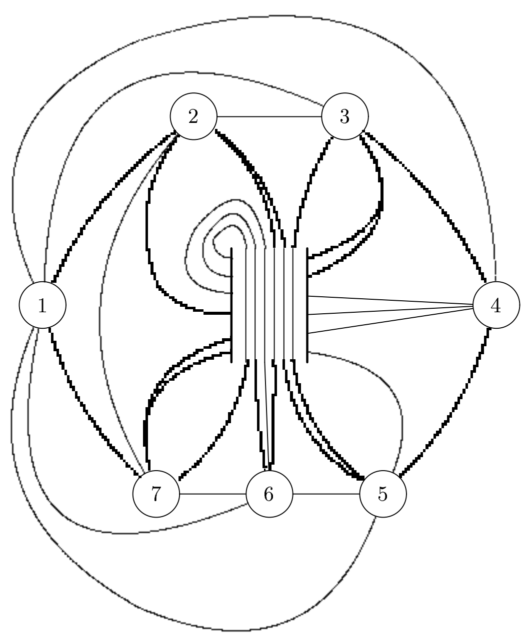


If a graph G is S_1-embeddable, then it can be drawn on any one of the above surfaces without crossing edges.



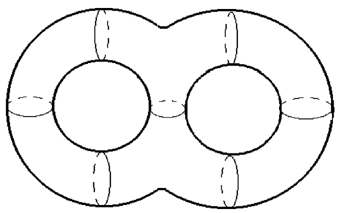


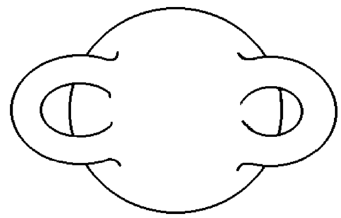
Example 10.2. The smallest non-planar graphs K_5 and $K_{3,3}$ have genus 1. Also the clique K_7 has genus 1. □



Genus

Let S_n ($n \geq 0$) be a sphere with n holes in it. The drawing of an S_4 can already be quite complicated, because we do not put any restrictions on the places of the holes (except that we must not tear the surface into disjoint parts). However, once again an S_n can be transformed (topologically) into a sphere with n handles.





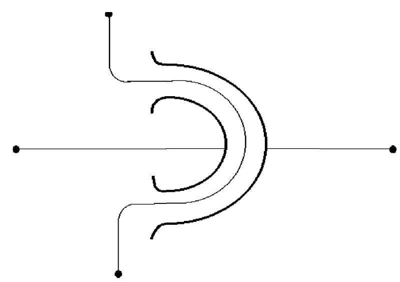
DEFINITION. We define the **genus** $g(G)$ of a graph G as the smallest integer n, for which G is S_n-embeddable. □

For planar graphs, we have $g(G) = 0$, and so $g(K_4) = 0$. For a clique K_5, $g(K_5) = 1$, since K_5 is nonplanar, but is embeddable on a torus. Also, $g(K_{3,3}) = 1$.

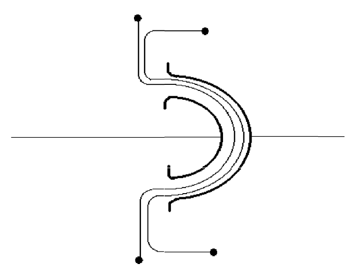
The next theorem states that any graph G can be embedded on some surface S_n with $n \geq 0$.

Theorem 10.3. *Every graph has a genus.*

This result has an easy intuitive verification. Indeed, consider a graph G and any of its plane (or sphere) drawing (possibly with many crossing edges) such that no three edges cross each other in the same point (such a drawing can be obtained). At each of these crossing points create a handle so that one of the edges goes below the handle and the other uses the handle to cross over the first one.



We should note that the above argument does not determine $g(G)$, only that G can be embedded on some S_n. However, clearly $g(G) \leq n$, and thus the genus $g(G)$ of G exists.
The same handle can be utilized by several edges.



Euler's formula with genus

The drawing of a planar graph G on a sphere has the advantage that the faces of the embedding are not divided into internal and external. The external face of G becomes an 'ordinary face' after G has been drawn on S_0.

In general, a *face* of an embedding of G on S_n (with $g(G) = n$) is a region of S_n surrounded by edges of G. Let again φ_G denote the number of faces of an embedding of G on S_n. We omit the proof of the next generalization of Euler's formula.

Theorem 10.4. *If G is a connected graph, then*

$$\nu_G - \varepsilon_G + \varphi_G = 2 - 2g(G)\,.$$

If G is a planar graph, then $g(G) = 0$, and the above formula is the Euler's formula for planar graphs.

DEFINITION. A face of an embedding $P(G)$ on a surface is a **2-cell**, if every simple closed curve (that does not intersect with itself) can be continuously deformed to a single point. □

A clique K_4 can be embedded on a torus such that it has a face that is not a 2-cell. But this is because $g(K_4) = 0$, and the genus of the torus is 1. We omit the proof of the general condition discovered by YOUNGS (1963):

Theorem 10.5. *The faces of an embedding of a connected graph G on a surface of genus $g(G)$ are 2-cells.*

Lemma 10.6. *For a connected G with $\nu_G \geq 3$ we have $3\varphi_G \leq 2\varepsilon_G$.*

Proof. If $\nu_G = 3$, then the claim is trivial. Assume thus that $\nu_G \geq 4$. In this case we need the knowledge that φ_G is counted on a surface that determines the genus of G (and on no surface with a larger genus). Now every face has a border of at least three edges, and, as before, every nonbridge is on the boundary of exactly two faces. □

Theorem 10.7. *For a connected G with $\nu_G \geq 3$,*

$$g(G) \geq \frac{1}{6}\varepsilon_G - \frac{1}{2}(\nu_G - 2)\,.$$

Proof. By the previous lemma, $3\varphi_G \leq 2\varepsilon_G$, and by the generalized Euler's formula, $\varphi_G = \varepsilon_G - \nu_G + 2 - 2g(G)$. Combining these we obtain that $3\varepsilon_G - 3\nu_G + 6 - 6g(G) \leq 2\varepsilon_G$, and the claim follows. □

By this theorem, we can compute lower bounds for the genus $g(G)$ without drawing any embeddings. As an example, let $G = K_8$. In this case $\nu_G = 8$, $\varepsilon_G = 28$, and so $g(G) \geq \frac{5}{3}$. Since the genus is always an integer, $g(G) \geq 2$. We deduce that K_8 cannot be embedded on the surface S_1 of the torus.

If H is a subgraph of G, then clearly $g(H) \leq g(G)$, since H is obtained from G by omitting vertices and edges. In particular,

Lemma 10.8. *For a graph G of order n, $g(G) \leq g(K_n)$.*

For the cliques K_n HEAWOOD (1890) proved a good lower bound.

Theorem 10.9. *If $n \geq 3$, then $g(K_n) \geq \frac{(n-3)(n-4)}{12}$.*

Proof. The number of edges in K_n is equal to $\varepsilon_G = \frac{1}{2}n(n-1)$. By Theorem 10.7, we obtain $g(K_n) \geq (1/6)\varepsilon_G - (1/2)(n-2) = (1/12)(n-3)(n-4)$. □

This result was dramatically improved by RINGEL AND YOUNGS in 1968. To state their result (without the long proof) we let $\lceil x \rceil$ denote the least integer k with $x \leq k$.

Theorem 10.10. *If $n \geq 3$, then $g(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$.*

Therefore $g(K_6) = \lceil 3 \cdot 2/12 \rceil = \lceil 1/2 \rceil = 1$. Also, $g(K_7) = 1$, but $g(K_8) = 2$.
By Theorem 10.10,

Theorem 10.11. *For all graphs G of order $n \geq 3$, $g(G) \leq \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$.*

Also, RINGEL (1965) has shown

Theorem 10.12. *For the complete bipartite graphs $g(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$.*

Chromatic numbers

For the planar graphs G, we noted that the proof of the 4-Colour Theorem, $\chi(G) \leq 4$, is extremely long and difficult. With this in mind, it is surprising that the generalization of the 4-Colour Theorem for genus ≥ 1 is much easier. HEAWOOD proved a hundred years ago:

Theorem 10.13. *If $g(G) = g \geq 1$, then $\chi(G) \leq \left[\frac{7+\sqrt{1+48g}}{2} \right]$.*

Notice that for $g = 0$ this theorem would be the 4-colour theorem. HEAWOOD proved it 'only' for $g \geq 1$.

Using the result of RINGEL AND YOUNGS and some elementary computations we can prove that the above theorem is the best possible.

Theorem 10.14. *For each $g \geq 1$, there exists a graph G with genus $g(G) = g$ so that*

$$\chi(G) = \left[\frac{7+\sqrt{1+48g}}{2} \right].$$

Hence if a nonplanar graph G can be embedded on a torus, then $g(G) = 1$, and $\chi(G) \leq [(7 + \sqrt{1+48g})/2] = 7$. Moreover, for $G = K_7$ we have that $\chi(K_7) = 7$ and $g(K_7) = 1$.

11 Switching of Graphs*

One can transform graphs into other graphs in numerous ways. As an example, $G \mapsto \overline{G}$ is an operation that transforms G into its complement graph. We introduce switching (also known as vertex-switching and Seidel switching), which provides a rather dynamic operation on graphs. Switching was originally defined in connection with a problem of finding equilateral n-tuples of points in elliptic geometry by VAN LINT AND SEIDEL (1966).

Switching modulo 2

DEFINITION. Let G be a graph, and let $A \subseteq V_G$ be a subset of vertices. The **switch** $G^A = (V_G, E^A)$ of G by A is the graph, where each edge that leaves A is changed to a nonedge, and each nonedge that leaves A is changed to an edge. The edges and nonedges of $G[A]$ and $G[V - A]$ are left intact. □

In the context of switching, it is more convenient to consider graphs as functions $G\colon E(V) \to \mathbb{Z}_2$, where $E(V) = \{uv \mid u, v \in V, u \neq v\}$ and $\mathbb{Z}_2 = \{0, 1\}$ is the additive group modulo 2. Here we can interpret

$$G(e) = \begin{cases} 1, & \text{if } e \in E\,, \\ 0, & \text{if } e \notin E\,. \end{cases}$$

Let $\Gamma(V)$ be the set of all graphs on the vertex set V.

DEFINITION. The **sum** of two graphs $G, H\colon E(V) \to \mathbb{Z}_2$, on the same set of vertices V, is defined as the graph $G + H$ such that $(G + H)(e) = G(e) + H(e)$. □

For the discrete graph $\overline{K}_V$ (with no edges), we adopt a simpler notation 0_V. Now $0_V(e) = 0$ for all $e \in E(V)$. Since for all graphs G, $G + G = 0_V$, we easily conclude

Theorem 11.1. *$\Gamma(V)$ is an Abelian group, where $G = -G$ for each graph G on V. The zero element of this group is the discrete graph 0_V.*

We rephrase the notion of switching by introducing (nonproper) 2-colourings.

DEFINITION. Any function $\sigma\colon V \to \mathbb{Z}_2$ is a **selector**. For a graph $G\colon E(V) \to \mathbb{Z}_2$ and a selector $\sigma\colon V \to \mathbb{Z}_2$, define a new graph G^σ, the **switch of** G **by** σ, such that for all $uv \in E(V)$,

$$G^\sigma(uv) = \sigma(u) + G(uv) - \sigma(v)\,. \tag{11.1}$$

□

Since $-a = a$ in $\mathbb{Z}_2$, the above definition can be rewritten as

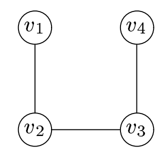
$$G^\sigma(uv) = \sigma(v) + G(uv) + \sigma(v)\,. \tag{11.2}$$

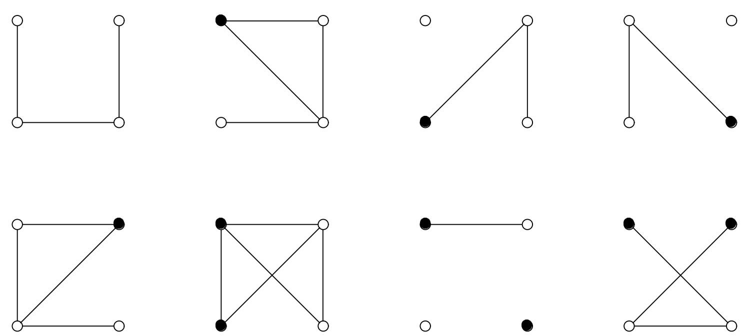
The general formulation (11.1) is stated because one is, anyway, tempted to replace $\mathbb{Z}_2$ by a general (Abelian or even nonabelian) group Γ. (This temptation could not be resisted by many authors since 1974.)

Theorem 11.2. *Let G be a graph, and $A \subseteq V_G$. Define $\sigma\colon V_G \to \mathbb{Z}_2$ by $\sigma(u) = 1$ for all $u \in A$ and $\sigma(u) = 0$ for all $u \notin A$. Then $G^A = G^\sigma$, and thus the above two definitions are equivalent.*

Proof. Exercise. □

Let $G = P_4$ be on the right. There are $2^{|V|} = 16$ selectors $\sigma\colon V \to \mathbb{Z}_2$, but some of them give the same switch G^σ. In fact, there are only 8 different (and only 5 nonisomorphic) switches G^σ. These have been drawn by putting a circle for a vertex with the value 0, and a black dot for a vertex with the value 1.





Switching classes

Definition. Let V be a given set of vertices. The set

$$[G] = \{G^\sigma \mid \sigma\colon V \to \mathbb{Z}_2\}$$

of graphs on V is called the **switching class** of the graph G (on V). □

For each selector $\sigma\colon V \to \mathbb{Z}_2$, we have

$$0_V^\sigma(uv) = \sigma(u) + \sigma(v)\,, \tag{11.3}$$

whenever $uv \in E(V)$. Also, for all bipartitions $V = A \cup B$, where A or $B = \overline{A}$ can be empty, denote by K_{AB} the complete bipartite graph with this bipartition (A, B):

$$K_{AB}(uv) = \begin{cases} 0, & \text{if } u, v \in A \text{ or } u, v \in B\,, \\ 1, & \text{otherwise}\,. \end{cases}$$

DEFINITION. For a selector $\sigma\colon V \to \mathbb{Z}_2$, let its **complement** be the selector defined for $v \in V$ by

$$\overline{\sigma}(v) = 1 - \sigma(v)\,.$$

For selectors $\sigma, \tau\colon V \to \mathbb{Z}_2$, their **sum** $\sigma + \tau$ is defined componentwise: for all $u \in V$,

$$(\sigma + \tau)(u) = \sigma(u) + \tau(u)\,.$$

□

Hence $\overline{\sigma}$ changes the values of σ. Clearly, $\sigma + \tau$ is a selector.

Elementary properties of switching

The following lemma states some basic properties of switching.

Lemma 11.3. *Let $G\colon E(V) \to \mathbb{Z}_2$, and $\sigma, \tau\colon V \to \mathbb{Z}_2$. Then*
- (i) $G^\sigma = G^{\overline{\sigma}}$,
- (ii) $(G^\sigma)^\tau = G^{\sigma+\tau}$,
- (iii) $G^\sigma + G^\tau = 0_V^{\sigma+\tau}$,
- (iv) $(G^\sigma)^\sigma = G$.

Proof. Let $uv \in E(V)$. Since the addition is modulo 2,

$$\begin{aligned}
G^{\overline{\sigma}}(uv) &= \overline{\sigma}(u) + G(uv) + \overline{\sigma}(v) = 1 - \sigma(u) + G(uv) + (1 - \sigma(v)) \\
&= \sigma(u) + G(uv) + \sigma(v) = G^\sigma(uv)\,, \\
(G^\sigma)^\tau(uv) &= \tau(u) + G^\sigma(uv) + \tau(v) = \tau(u) + \sigma(u) + G(uv) + \sigma(v) + \tau(v) \\
&= (\sigma + \tau)(u) + G(uv) + (\sigma + \tau)(v) = G^{\sigma+\tau}(uv)\,, \\
(G^\sigma + G^\tau)(uv) &= G^\sigma(uv) + G^\tau(uv) = \sigma(u) + G(uv) + \sigma(v) + \tau(u) + G(uv) + \tau(v) \\
&= (\sigma + \tau)(u) + (\sigma + \tau)(v) = 0_V^{\sigma+\tau}(uv)\,.
\end{aligned}$$

Case (iv) follows from (ii), since $(\sigma + \sigma)(u) = 0$ for all $x \in V$. □

We give now a more detailed version of Lemma 11.3(iii).

Lemma 11.4. *Let $G\colon E(V) \to \mathbb{Z}_2$ be a graph and $\sigma\colon V \to \mathbb{Z}_2$ a selector. Then*

$$G + G^\sigma = K_{OI}\,,$$

where $O = \sigma^{-1}(0)$ and $I = \sigma^{-1}(1)$.

Proof. If $u, v \in O$ or $u, v \in I$, then $\sigma(u) = \sigma(v) = 0$; otherwise $\sigma(u) + \sigma(v) = 1$. Therefore

$$(G + G^\sigma)(uv) = G(uv) + \sigma(u) + G(uv) + \sigma(v) = \sigma(u) + \sigma(v) = K_{OI}(uv)\,,$$

for all $uv \in E(V)$. □

The first claim of the next theorem states that a switching class $[G]$ is independent of its 'generator'. The second claim gives a characterization of the complete bipartite graphs in terms of switching.

Theorem 11.5. *Let V be a set of vertices. Then*

(i) $[G] = [G^\sigma]$ *for all* $G\colon E(V) \to \mathbb{Z}_2$ *and* $\sigma\colon V \to \mathbb{Z}_2$.
(ii) $[0_V] = \{K_{AB} \mid A \cup B = V$ *a bipartition*$\}$.

Proof. Case (i) follows from Lemma 11.3(iv), since if $G\colon E(V) \to \mathbb{Z}_2$ and $\sigma\colon V \to \mathbb{Z}_2$, then $(G^\sigma)^\sigma = G$, and hence $G \in [G^\sigma]$.

For (ii), observe that if $G \in [0_V]$, then there exists a selector σ such that $G = 0_V^\sigma$. Let $O = \sigma^{-1}(0)$ and $I = \sigma^{-1}(1)$. Now for all $uv \in E(V)$, $G(uv) = 0_V^\sigma(uv) = \sigma(u) + \sigma(v)$, and hence $G(uv) = 0$ if and only if $u, v \in I$ or $u, v \in O$. Consequently, $G = K_{OI}$.

On the other hand, if $G = K_{AB}$ is a complete (A, B)-bipartite graph, then define σ by $\sigma(u) = 1$ for $u \in A$ and $\sigma(v) = 0$ for $v \in B$. Now, for all $uv \in E(V)$, $G^\sigma(uv) = \sigma(u) + G(uv) + \sigma(v) = 0$, and thus $G^\sigma = 0_V$. This proves (ii). □

As the following theorem states, the class $[0_V]$ has a central role among the switching classes. Indeed, all other switching classes are cosets of $[0_V]$ in the group $\Gamma(V)$:

Theorem 11.6. *Let V be a set of vertices. Then for each G on V,*

$$[G] = G + [0_V] = \{G + K_{AB} \mid A \cup B = V \text{ a bipartition}\}.$$

Proof. Note first that if $G, H \in [0_V]$, then by Lemma 11.3(iii), also $G + H \in [0_V]$, which shows that $[0_V]$ is a subgroup of $\Gamma(V)$.

By Lemma 11.4, if σ is a selector, then $G + G^\sigma = K_{OI}$ in $\Gamma(V)$, where again $O = \sigma^{-1}(0)$ and $I = \sigma^{-1}(1)$, and thus $G^\sigma = G + K_{OI}$. On the other hand, if $H = G + K_{AB}$, then $H = G^\sigma$, where $\sigma(u) = 1$ for all $u \in A$ and $\sigma(v) = 0$ for all $v \in B$. This proves the claim. □

A local characterization

We prove a theorem of SEIDEL (1976) stating that one can determine locally whether or not a graph H belongs to a switching class $[G]$. For a graph $G\colon E(V) \to \mathbb{Z}_2$ and a subset $A \subseteq V$, we set

$$G(A) = \sum_{uv \in E(A)} G(uv) \qquad (\text{ in } \mathbb{Z}_2).$$

Hence $G(A)$ is the parity of the number of edges of G in the subgraph $G[A]$.

Theorem 11.7. *Let $G\colon E(V) \to \mathbb{Z}_2$ be a graph. Then $H \in [G]$ if and only if for all subsets A of 3 vertices, $G[A]$ and $H[A]$ have the same parity of edges: $G(A) = H(A)$.*

Proof. Assume first that $H = G^\sigma$ for a selector σ, and let $A = \{u, v, w\}$ consist of different vertices. We have

$$\begin{aligned} G^\sigma(A) &= G^\sigma(uv) + G^\sigma(vw) + G^\sigma(wu) \\ &= 2 \cdot \sigma(u) + G(uv) + 2 \cdot \sigma(v) + G(vw) + 2 \cdot \sigma(w) + G(wu) \\ &= G(uv) + G(vw) + G(wu) = G(A) . \end{aligned}$$

In the converse direction, suppose that $G(A) = H(A)$ for all 3-element subsets A. Fix a vertex $u_0 \in V$, and define the selectors σ and τ by

$$\sigma(u_0) = 0 = \tau(u_0) \ , \quad \sigma(v) = G(u_0 v) \quad \text{and} \quad \tau(v) = H(u_0 v)$$

for all $v \neq u_0$. Consequently, $G^\sigma(u_0 v) = 0$ and $H^\tau(u_0 v) = 0$ for all $v \neq u_0$. Moreover, by the first part of the proof, for all 3-element subsets $A = \{u_0, v, w\}$, $H(A) = H^\tau(A)$ and $G(A) = G^\sigma(A)$, where

$$H^\tau(vw) = H^\tau(A) = G^\sigma(A) = G^\sigma(vw) .$$

Therefore $H^\tau = G^\sigma$, and the claim follows, since $[H] = [H^\tau] = [G^\sigma] = [G]$ using Theorem 11.5(i). □

Lemma 11.8. *Let $(v_1, v_2, \dots, v_k)$ be an ordered sequence of distinct vertices of V. For each $G = (V, E)$ and $\sigma \colon V \to \mathbb{Z}_2$, the parity of edges in $\{v_1 v_2, \dots, v_{k-1} v_k, v_k v_1\}$ is the same for G and G^σ.*

Proof. Exercise. □

Eulerian graphs in switching classes

As we know, G is Eulerian if and only if it is connected and has even degrees. So, a graph G has even degrees if and only if its components are Eulerian graphs.

Lemma 11.9. *If both G and H have even degrees, then so does their sum $G + H$.*

Proof. Exercise. □

The following result is due to Seidel (1976).

Theorem 11.10. *Let ν_G be odd. Then each switching class $[G]$ on V contains a unique graph with even degrees.*

Proof. Consider $G \colon E(V) \to \mathbb{Z}_2$ and define $\sigma \colon V \to \mathbb{Z}_2$ such that

$$\sigma(u) = \begin{cases} 1 & \text{if } d_G(u) \text{ is odd,} \\ 0 & \text{if } d_G(u) \text{ is even.} \end{cases}$$

By the handshaking lemma, G has an odd number of vertices with an even degree, since $|V|$ is assumed to be odd. For each $u \in V$, denote by $d_s(u)$ ($d_o(u)$, respectively) the number of its neighbours that have the same (opposite, respectively) parity as their degrees, and let $n_o(u)$ denote the number of all vertices with the opposite degree parity in G as u. Then $d_G(u) = d_s(u) + d_o(u)$. In G^σ we have

$$\begin{aligned} d_{G^\sigma}(u) &= d_s(u) + (n_o(u) - d_o(u)) = (d_G(u) - d_o(u)) + n_o(u) - d_o(u) \\ &= d_G(u) - 2d_o(u) + n_o(u) \equiv d_G(u) + n_o(u) \equiv 0 \pmod 2, \end{aligned}$$

since $d_G(u)$ and $n_o(u)$ have the same parity. We conclude that G^σ has even degrees.

The discrete graph 0_V has even degrees, and if G and H do, so does their sum $G + H$ by the previous lemma. Consequently, if both G and G^σ (with $G \neq G^\sigma$) have even degrees for some σ, then so does $K_{OI} = G + G^\sigma$, where again $O = \sigma^{-1}(0)$ and $I = \sigma^{-1}(1)$. Now, $O \neq V$ or $I \neq V$ (since $G^\sigma \neq G$), and therefore if K_{OI} has even degrees, then both $|O|$ and $|I|$ are even, and thus $|V| = |O| + |I|$ is even. This shows that $[G]$ contains a unique graph with even degrees, when $|V|$ is odd. □

For even ν_G, the above result does not hold. In this case, if a switching class contains a graph G with even degrees, then exactly half of $[G]$ share this property with G.

Hamiltonian graphs

We prove a theorem stating that each switching class $[G]$ on V with $\nu_G \geq 3$ contains a Hamiltonian graph except when $[G] = [0_V]$ for an odd $|V|$.

Lemma 11.11. *Let G be a graph on V, $|V| = n \geq 2$, and let $H \in [G]$ be such that H has the maximum number of edges. Then $\delta(H) \geq (n-1)/2$.*

Proof. Assume that $d_H(v) < (n-1)/2$, and let $\sigma(v) = 1$ and $\sigma(u) = 0$ for $u \neq v$. Then $d_{H^\sigma}(v) = n - 1 - d_H(v)$, and H^σ has more edges than H, contradicting the assumption on maximality. □

If n is even, then the above lemma states that $\delta(H) \geq n/2$, which, by Dirac's theorem, gives

Theorem 11.12. *If $|V| \geq 4$ is even, then for all graphs G on V, there exists a Hamiltonian graph G^σ in its switching class.*

We consider then the odd case.

Lemma 11.13. *Let $|V|$ be odd. The switching class $[0_V]$ (of all complete bipartite graphs on V) does not contain a Hamiltonian graph.*

Proof. Let $G = 0_V^\sigma \in [0_V]$ be Hamiltonian, and let $u_0 \longrightarrow u_1 \longrightarrow \ldots \longrightarrow u_{n-1} \longrightarrow u_0$ be a Hamilton cycle of G. Since $0_V(e) = 0$ for all edges e, σ must satisfy the condition: $\sigma(u_i) \neq \sigma(u_{i+1(n)})$ for all i. This is possible only if n is even. □

Lemma 11.14. *Let G be a graph on V.*

(i) *For each $v \in V$, there exists a σ such that $vu \in E_{G^\sigma}$ for all $u \neq v$.*
(ii) *If $S = (v_0, v_1, \dots, v_{k-1})$ is any sequence of distinct vertices of V, then there exists a σ such that $v_0 \longrightarrow v_1 \longrightarrow \dots \longrightarrow v_{k-1}$ is a path in G^σ.*

Proof. Exercise. □

We derive a rather simple criterion due to KRATOSVÍL AND PELANT (1993) for the existence of Hamiltonian graphs in switching classes.

Theorem 11.15. *Let $|V| = n \geq 3$. A switching class $[G]$ on V contains a Hamiltonian graph if and only if $[G] \neq [0_V]$ for odd n.*

Proof. The case $n = 3$ is clear, and we have already shown the claim for even n. Assume thus that n is odd. Also, the claim is trivial for $[K_n]$ since K_n is Hamiltonian for $n \geq 3$. We can thus assume that $n \geq 5$, and that $[G]$ does not contain 0_V or K_n.

By Lemma 11.14, we may assume that G has a vertex v such that $vu \in E_G$ for all $u \neq v$. Since $G \neq K_n$, and $G{-}v \neq 0_V$ (otherwise $0_V \in [G]$), there is a 3-element subset $A = \{v_1, v_2, v_3\} \subseteq V - \{v\}$ such that $v_1v_2 \in E_G$ and $v_2v_3 \notin E_G$ (here v_1v_3 does not matter). Let $V = \{v, v_1, v_2, v_3, \dots, v_{n-1}\}$, and Consider the sequences

$$S_1: \quad v_1v, vv_2, v_2v_3, \dots, v_{n-2}v_{n-1}, v_{n-1}v_1\,,$$
$$S_2: \quad v_1v_2, v_2v, vv_3, \dots, v_{n-2}v_{n-1}, v_{n-1}v_1\,.$$

These have a different parity of edges, and thus either one of these, say S_1, has an odd number of edges. By Lemma 11.8, each G^σ has an odd number of edges in the sequence S_1. In particular, this holds for the graph G^σ, where $P_1 \colon v_1 \longrightarrow v \longrightarrow v_2 \longrightarrow \dots \longrightarrow v_{n-1}$ is a Hamiltonian path. The existence of such a σ is guaranteed by Lemma 11.14. Since n is odd, the Hamiltonian path P_1 has an even number of edges. But, by the above, the corresponding cycle S_1 has an odd number of edges, which means that also $v_{n-1}v_1$ is an edge in G^σ, and therefore $P_1 \longrightarrow v_1$ is a Hamiltonian cycle in G^σ. □

The Hamiltonian graphs in $[G]$ need not be unique. For this, one needs only to consider the cycle $G = C_5$.

A switching class can contain more than one tree as seen from our example for P_4. However, the trees in a switching class are all isomorphic to each other, and, in fact, a switching class can contain two trees only in some special cases. This result is due to HAGE AND TH (1996).

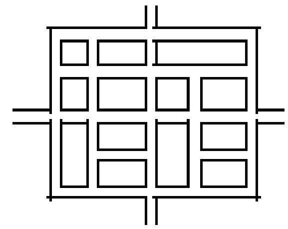
Theorem 11.16. *Each switching class contains at most one tree up to isomorphism.*

For the theory of switching and its generalizations to mappings $\alpha \colon E(V) \to \Gamma$ for arbitrary groups, see

A. EHRENFEUCHT, T. HARJU AND G. ROZENBERG, "Theory of 2-Structures. A framework for decomposition and transformation of graphs." World Scientific, in print.

12 Directed Graphs

In some problems the relation between the objects is not symmetric. For these cases we need directed graphs, where the edges are oriented from one vertex to another.
As an example consider a map of a small town. Can you make the streets one-way, and still be able to drive from one house to another (or exit the town)?



Definitions

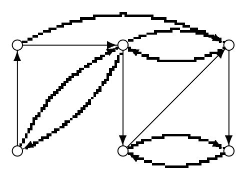
Definition. A **digraph** (or a **directed graph**) $D = (V_D, E_D)$ consists of the vertices V_D and (directed) edges $E_D \subseteq V_D \times V_D$ (without loops vv). We still write uv for (u, v), but note that now $uv \neq vu$. For each pair $e = uv$ define the **inverse** of e as $e^{-1} = vu$ $(= (v, u))$. For an edge $e = uv$, u and v are called the **tail** and the **head** of e, respectively. □

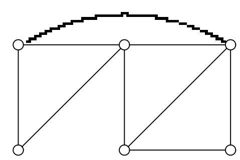
Note that $e \in E_D$ does *not* imply $e^{-1} \in E_D$.

Definition. Let D be a digraph. Then A is its

- **subdigraph**, if $V_A \subseteq V_D$ and $E_A \subseteq E_D$,
- **induced subdigraph**, $A = D[X]$, if $V_A = X$ and $E_A = E_D \cap (X \times X)$.

The **underlying graph** $U(D)$ of a digraph D is the graph on V_D such that if $e \in E_D$, then the undirected edge with the same ends is in $U(D)$.





A digraph D is an **orientation** of a graph G, if $G = U(D)$ and $e \in E_D$ implies $e^{-1} \notin E_D$. In this case, D is said to be an **oriented graph**. □

Definition. Let D be a digraph. A walk $W = e_1 e_2 \dots e_k \colon u \xrightarrow{\star} v$ of $U(D)$ is a **directed walk**, if $e_i \in E_D$ for all $i \in [1, k]$. Similarly, we define **directed paths** and **directed cycles** as directed walks and closed directed walks without repetitions of vertices.

The digraph D is **di-connected**, if, for all $u \neq v$, there exist directed paths $u \xrightarrow{\star} v$ and $v \xrightarrow{\star} u$. The maximal induced di-connected subdigraphs are the **di-components** of D. □

Note that a graph $G = U(D)$ might be connected, although the digraph D is not di-connected.

DEFINITION. The **indegree** and the **outdegree** of a vertex are defined as follows

$$d_D^I(v) = |\{e \in E_D \mid e = xv\}|, \qquad d_D^O(v) = |\{e \in E_D \mid e = vx\}|.$$

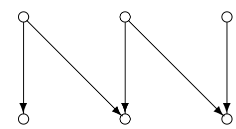
□

We have the following **handshaking lemma**. Here you offer and accept a handshake.

Lemma 12.1. $\sum_{v \in V_D} d_D^I(v) = |E_D| = \sum_{v \in V_D} d_D^O(v)$.

Directed paths

The relationship between paths and directed paths is in general rather complicated. This digraph has a path of length five, but its directed paths are of length one.



ROY (1967) and GALLAI (1968) proved a connection between the lengths of directed paths and the chromatic number $\chi(D) = \chi(U(D))$.

Theorem 12.2. *A digraph D has a directed path of length $\chi(D) - 1$.*

Proof. Let $A \subseteq E_D$ be a minimal set of edges such that the subdigraph $D{-}A$ contains no directed cycles. Let k be the length of the longest directed path in $D{-}A$.

For each vertex $v \in V_D$, assign a colour $\alpha(v) = i$, if a longest directed path from v has length $i - 1$ in $D{-}A$. Here $1 \le i \le k + 1$.

First we observe that if $P = e_1 e_2 \dots e_r$ $(r \ge 1)$ is any directed path $u \xrightarrow{\star} v$ in $D{-}A$, then $\alpha(u) \neq \alpha(v)$. Indeed, if $\alpha(v) = i$, then there exists a directed path $Q \colon v \xrightarrow{\star} w$ of length $i - 1$, and PQ is a directed path, since $D{-}A$ does not contain directed cycles. Since $PQ \colon u \xrightarrow{\star} w$, $\alpha(u) \neq i = \alpha(v)$. In particular, if $e = uv \in E_D - A$, then $\alpha(u) \neq \alpha(v)$.

Consider then an edge $e = vu \in A$. By the minimality of A, $(D{-}A) + e$ contains a directed cycle $C \colon u \xrightarrow{\star} v \longrightarrow u$, where the part $u \xrightarrow{\star} v$ is a directed path in $D{-}A$, and hence $\alpha(u) \neq \alpha(v)$. This shows that α is a proper colouring of $U(D)$, and therefore $\chi(D) \le k + 1$, that is, $k \ge \chi(D) - 1$. □

The bound $\chi(D) - 1$ is the best possible in the following sense:

Theorem 12.3. *Every graph G has an orientation D, where the longest directed paths have lengths $\chi(G) - 1$.*

Proof. Let $k = \chi(G)$ and let α be a proper k-colouring of G. As usual the set of colours is $[1, k]$. We orient each edge $uv \in E_G$ by setting $uv \in E_D$, if $\alpha(u) < \alpha(v)$. Clearly, the so obtained orientation D has no directed paths of length $\ge k - 1$. □

DEFINITION. An orientation D of an undirected graph G is **acyclic**, if it has no directed cycles. Let $a(G)$ be the number of acyclic orientations of G. □

The next result of STANLEY (1973) is rather strange, since $\chi_G(-1)$ measures the number of proper colourings of G using -1 colours!

Theorem 12.4. *Let G be a graph of order n. Then the number of the acyclic orientations of G is*

$$a(G) = (-1)^n \chi_G(-1)\,,$$

where χ_G is the chromatic polynomial of G.

Proof. The proof by induction on ε_G. First, if G is discrete, then $\chi_G(k) = k^n$, and $a(G) = 1 = (-1)^n(-1)^n = (-1)^n\chi_G(-1)$ as required.

Now $\chi_G(k)$ is a polynomial that satisfies the recurrence $\chi_G(k) = \chi_{G-e}(k) - \chi_{G*e}(k)$. To prove the claim, we show that $a(G)$ satisfies the same recurrence.

Indeed, if

$$a(G) = a(G-e) + a(G*e) \tag{12.1}$$

then, by the induction hypothesis,

$$a(G) = (-1)^n\chi_{G-e}(-1) + (-1)^{n-1}\chi_{G*e}(-1) = (-1)^n\chi_G(-1)\,.$$

For (12.1), we observe that every acyclic orientation of G gives an acyclic orientation of $G-e$. On the other hand, if D is an acyclic orientation of $G-e$ for $e = uv$, it extends to an acyclic orientation of G by putting $e_1\colon u \to v$ or $e_2\colon v \to u$. Indeed, if D has no directed path $u \xrightarrow{\star} v$, we choose e_2, and if D has no directed path $v \xrightarrow{\star} u$, we choose e_1. Note that since D is acyclic, it cannot have both ways $u \xrightarrow{\star} v$ and $v \xrightarrow{\star} u$.

We conclude that $a(G) = a(G-e) + b$, where b is the number of acyclic orientations D of $G-e$ that extend in both ways e_1 and e_2. The acyclic orientations D that extend in both ways are exactly those that contain

(12.2) neither $u \xrightarrow{\star} v$ nor $v \xrightarrow{\star} u$ as a directed path.

Each acyclic orientation of $G*e$ corresponds in a natural way to an acyclic orientation D of $G-e$ that satisfies (12.2). Therefore $b = a(G*e)$, and the proof is completed. □

One-way traffic

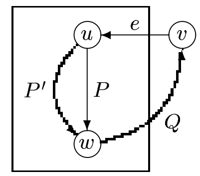
Every graph can be oriented, but the result may not be di-connected. In the **one-way traffic problem** the resulting orientation should be di-connected, for otherwise someone is not able to drive home. ROBBINS' theorem (1939) solves this problem.

DEFINITION. A graph G is **di-orientable**, if there is a di-connected oriented graph D such that $G = U(D)$. □

Theorem 12.5. *A connected graph G is di-orientable if and only if G has no bridges.*

Proof. If G has a bridge e, then any orientation of G has at least two di-components (both sides of the bridge).

Suppose then that G has no bridges. Hence G has a cycle C, and a cycle is always di-orientable. Let then H be a maximal subgraph of G that has a di-orientation D_H. If here $H = G$, then we are done.



Otherwise, there exists an edge $e = vu \in E_G$ such that $u \in V_H$ but $v \notin V_H$ (because G is connected). The edge e is not a bridge and thus there exists a cycle

$$C' = ePQ\colon v \longrightarrow u \overset{\star}{\longrightarrow} w \overset{\star}{\longrightarrow} v$$

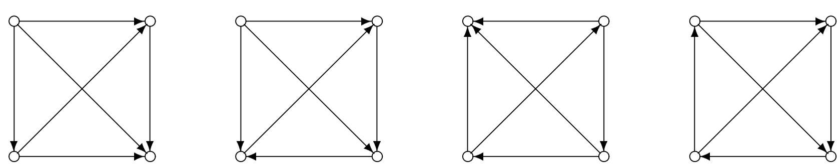
in G, where w is the last vertex inside H.

Now in the di-orientation D_H of H there is a directed path $P'\colon u \overset{\star}{\longrightarrow} w$. When we orient $e\colon v \longrightarrow u$ and the edges of Q in the direction $Q\colon w \overset{\star}{\longrightarrow} v$, we obtain a directed cycle $eP'Q\colon v \longrightarrow u \overset{\star}{\longrightarrow} w \overset{\star}{\longrightarrow} v$. In conclusion, the subgraph $G[V_H \cup V_C]$ has a di-orientation, which contradicts the maximality assumption on H. This proves the claim. □

Tournaments

DEFINITION. A **tournament** T is an orientation of a clique. □

There are four tournaments of four vertices that are not isomorphic with each other. (Isomorphism of directed graphs is defined in the obvious way.)



The following is due to RÉDEI (1934).

Theorem 12.6. *Every tournament has a directed Hamilton path.*

Proof. The chromatic number of K_n is $\chi(K_n) = n$, and hence by Theorem 12.2, a tournament T of order n has a directed path of length $n - 1$. This is then a directed Hamilton path visiting each vertex once. □

The vertices of a tournament can be easily reached from one vertex (sometimes called the **king**).

Theorem 12.7. *Let v be a vertex of a tournament T of maximum outdegree. Then for all u, there is a directed path $v \xrightarrow{\star} u$ of length at most two.*

Proof. Let T be an orientation of K_n, and let $d_T^O(v) = d$ be the maximum outdegree in T. Suppose that there exists an x, for which the directed distance from v to x is at least three. It follows that $xv \in E_T$ and $xu \in E_T$ for all u with $vu \in E_T$. But there are d vertices in $A = \{y \mid vy \in E\}$, and thus $d+1$ vertices in $\{y \mid xy \in E\} = A \cup \{v\}$. It follows that the outdegree of x is $d+1$, which contradicts the maximality assumption made for v. □

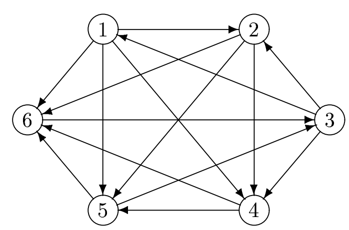
Research problem: Ádám's conjecture states that *in every digraph D with a directed cycle there exists an edge uv the reversal of which decreases the number of directed cycles.* Here the new digraph has the edge vu instead of uv.

Example 12.8. Consider a tournament of n teams that play once against each other, and suppose that each game has a winner. The situation can be presented as a tournament, where the vertices correspond to the teams v_i, and there is an edge v_iv_j, if v_i won v_j in their mutual game.

Let us say that a team v is *a* **winner** (there may be more than one winner), if v comes out with the most victories in the tournament. Theorem 12.7 states that a winner v either defeated a team u or v defeated a team that defeated u.

A **ranking** of a tournament is a linear ordering of the teams $v_{i_1} > v_{i_2} > \cdots > v_{i_n}$ that should reflect the scoring of the teams. One way of ranking a tournament could be by a Hamilton path: the ordering can be obtained from a directed Hamilton path $P\colon v_{i_1} \longrightarrow v_{i_2} \longrightarrow \dots \longrightarrow v_{i_n}$. However, a tournament may have several directed Hamilton paths, and some of these may do unjust for the 'real' winner.

Consider a tournament of six teams $1, 2, \dots, 6$, and let T be the scoring digraph as in the figure. Here $1 \longrightarrow 2 \longrightarrow 4 \longrightarrow 5 \longrightarrow 6 \longrightarrow 3$ is a directed Hamilton path, but this extends to a directed Hamilton cycle (by adding $3 \longrightarrow 1$)! So for every team there is a Hamilton path, where it is a winner, and in another, it is a looser.



Let $s_1(j) = d_T^O(j)$ be the **winning number** of the team j (the number of teams beaten by j). In the above tournament,

$$s_1(1) = 4,\ s_1(2) = 3,\ s_1(3) = 3,\ s_1(4) = 2,\ s_1(5) = 2,\ s_1(6) = 1\,.$$

So, is team 1 the winner? If so, is 2 or 3 next? Define the **second-level scoring** for each team by

$$s_2(j) = \sum_{ji \in E_T} s_1(i)\,.$$

This tells us how good teams j beat. In our example, we have

$$s_2(1) = 8,\ s_2(2) = 5,\ s_2(3) = 9,\ s_2(4) = 3,\ s_2(5) = 4,\ s_2(6) = 3\,.$$

Now, it seems that 3 is the winner,but 4 and 6 have the same score. We continue by defining inductively the m**th-level scoring** by

$$s_m(j) = \sum_{ji \in E_T} s_{m-1}(i)\,.$$

It can be proved (using matrix methods) that for a di-connected tournament with at least four teams, *the level scorings will eventually stabilize in a ranking of the tournament*: there exits an m for which the mth-level scoring gives the same ordering as do the $(m+k)$th-level scorings for all $k \geq 1$. If T is not di-connected, then the level scoring should be carried out with respect to the di-components.

In our example the level scoring gives $1 \longrightarrow 3 \longrightarrow 2 \longrightarrow 5 \longrightarrow 4 \longrightarrow 6$ as the ranking of the tournament. □

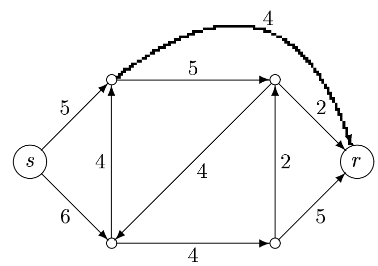
13 Flows in Networks

Various transportation networks or water pipelines are conveniently represented by weighted directed graphs. These networks possess some additional requirements. Goods are transported from specific places (warehouses) to final locations (marketing places) through a network of roads. In modelling a transportation network by a digraph, we must make sure that the number of goods remains the same at each crossing of the roads.

Flows

DEFINITION. A **network** N consists of

- an **underlying digraph** $D = (V, E)$,
- two distinct vertices s and r, called the **source** and the **sink** of N, and
- a **capacity function** $\alpha\colon V \times V \to \mathbb{R}_+$ (nonnegative real numbers), for which $\alpha(e) = 0$, if $e \notin E$.



Denote $V_N = V$, $E_N = E$, and $I = V_N - \{r, s\}$. □

Let $A \subseteq V_N$ be a set of vertices, and $f\colon V_N \times V_N \to \mathbb{R}$ any function such that $f(e) = 0$, if $e \notin E_N$. We adopt the following notations:

$$[A, \overline{A}] = \{e \in E_D \mid e = uv,\ u \in A,\ v \notin A\},$$

$$f^+(A) = \sum_{e \in [A, \overline{A}]} f(e) \quad \text{and} \quad f^-(A) = \sum_{e \in [\overline{A}, A]} f(e)\,.$$

In particular,

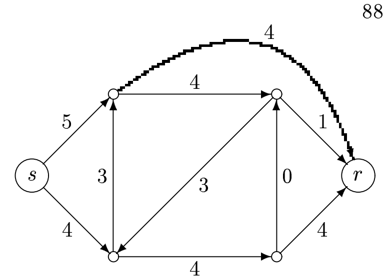
$$f^+(u) = \sum_{v \in V_N} f(uv) \quad \text{and} \quad f^-(u) = \sum_{v \in V_N} f(vu)\,.$$

DEFINITION. A **flow** in a network N is a function $f\colon V_N \times V_N \to \mathbb{R}_+$ such that

$$\begin{aligned} &0 \le f(e) \le \alpha(e) && \text{for all } \ e\,, \\ &f^-(v) = f^+(v) && \text{for all } \ v \in I\,. \end{aligned}$$

□

The value $f(e)$ is the rate at which transportation actually happens along the channel e which has the maximum capacity $\alpha(e)$. The second condition states that there should be no loss.
If $N = (D, s, r, \alpha)$ is a network of water pipes, then the value $\alpha(e)$ gives the capacity ($x\ m^3$/min) of the pipe e.



A flow f in N is something that the network can handle. *E.g.*, in the above figure the source should not try to feed the network the full capacity ($11\ m^3$/min) of its pipes, because the junctions cannot handle this much water.

Definition. Every network N has a **zero flow** defined by $f(e) = 0$ for all e. For a flow f and each subset $A \subseteq V_N$, define the **resultant flow from** A and the **value** of f as the numbers

$$\text{val}(f_A) = f^+(A) - f^-(A) \qquad \text{and} \qquad \text{val}(f) = \text{val}(f_s)\ (= f^+(s) - f^-(s))\,.$$

A flow f of a network N is a **maximum flow**, if there does not exist any flow f' such that $\text{val}(f) < \text{val}(f')$. □

The value $\text{val}(f)$ of a flow is the overall number of goods that are (to be) transported through the network from the source to the sink. In the above example, $\text{val}(f) = 9$.

Our first claim is clear.

Lemma 13.1. *Let $N = (D, s, r, \alpha)$ be a network with a flow f.*

(i) *If $A \subseteq I$, then* $\text{val}(f_A) = 0$.
(ii) $\text{val}(f) = -\text{val}(f_r)$.

Proof. Let $A \subseteq I$. Then

$$\begin{aligned}
\text{val}(f_A) = f^+(A) - f^-(A) &= \sum_{e\in[A,\overline{A}]} f(e) - \sum_{e\in[\overline{A},A]} f(e) \\
&= \sum_{u\in A, v\notin A} f(uv) - \sum_{u\in A, v\notin A} f(vu) \\
&= \sum_{u\in A}\Big(\sum_{v\in V_N} f(uv) - \sum_{v\in A} f(uv)\Big) - \sum_{u\in A}\Big(\sum_{v\in V_N} f(vu) - \sum_{v\in A} f(vu)\Big) \\
&= \sum_{u\in A}\Big(\sum_{v\in V_N} f(uv) - \sum_{v\in V_N} f(vu)\Big) - \sum_{u\in A}\sum_{v\in A} f(uv) + \sum_{u\in A}\sum_{v\in A} f(vu) \\
&= \sum_{u\in A}(f^+(u) - f^-(u)) = 0\,.
\end{aligned}$$

The second claim is also clear. □

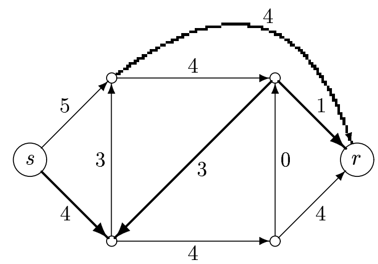
Improvable flows

Let f be a flow in a network N, and let $P = e_1 e_2 \dots e_n$ be an *undirected* path in N where an edge e_i is **along** P, if $e_i = v_i v_{i+1} \in E_N$, and **against** P, if $e_i = v_{i+1} v_i \in E_N$.

We define a nonnegative number $\iota(P)$ for P as follows:

$$\iota(P) = \min_{e_i} \iota(e), \qquad \text{where } \iota(e) = \begin{cases} \alpha(e) - f(e) & \text{if } e \text{ is along } P\,, \\ f(e) & \text{if } e \text{ is against } P. \end{cases}$$

DEFINITION. Let f be a flow in a network N. A path $P\colon s \xrightarrow{\star} r$ is (f-)**improvable**, if $\iota(P) > 0$. □

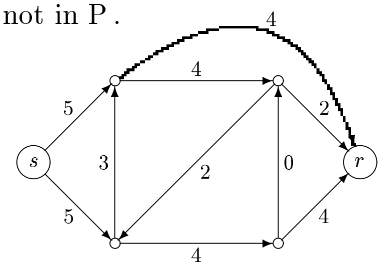


Lemma 13.2. *Let N be a network. If f is a maximum flow of N, then it has no improvable paths.*

Proof. Define

$$f'(e) = \begin{cases} f(e) + \iota(P) & \text{if } e \text{ is along P}\,, \\ f(e) - \iota(P) & \text{if } e \text{ is against P}\,, \\ f(e) & \text{if } e \text{ is not in P}\,. \end{cases}$$

Then f' is a flow, since at each intermediate vertex $v \in I$, we have $(f')^-(v) = (f')^+(v)$, and the capacities of the edges are not exceeded.
Now $\text{val}(f') = \text{val}(f) + \iota(P)$, since P has exactly one edge $sv \in E_N$ for the source s. Hence, if $\iota(P) > 0$, then we can improve the flow.



□

Max-Flow Min-Cut Theorem

DEFINITION. Let $N = (D, s, r, \alpha)$ be a network. For a subset $S \subset V_N$ with $s \in S$ and $r \notin S$, let the **cut** by S be

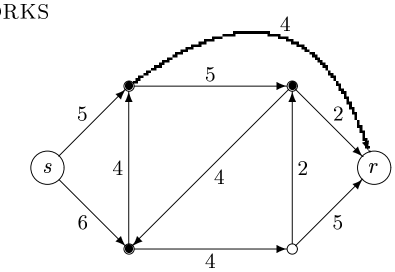
$$[S] = [S, \overline{S}] \quad (= \{uv \in E_N \mid u \in S, v \notin S\})\,.$$

The **capacity** of the cut $[S]$ is the sum

$$\alpha[S] = \alpha^+(S) = \sum_{e \in [S]} \alpha(e)\,.$$

A cut $[S]$ is a **minimum cut**, if there is no cut $[R]$ with $\alpha[R] < \alpha[S]$. □

The capacity of the cut for the indicated vertices is equal to 10.



Lemma 13.3. *For a flow f and a cut $[S]$ of N,* $\mathrm{val}(f) = \mathrm{val}(f_S) = f^+(S) - f^-(S)$.

Proof. By definitions, $\mathrm{val}(f) = \mathrm{val}(f_s)$, and $\mathrm{val}(f_v) = 0$, if $v \in S - \{s\}$. Let $S_I = S - \{s\}$. Now $\mathrm{val}(S_I) = 0$ (since $S_I \subseteq I$), and

$$\begin{aligned}\mathrm{val}(f_S) &= \mathrm{val}(f_s) + \mathrm{val}(f_{S_I}) - \sum_{v \in S_I} f(sv) + \sum_{v \in S_I} f(vs) - \sum_{v \in S_I} f(sv) + \sum_{v \in S_I} f(vs) \\ &= \mathrm{val}(f_s) = \mathrm{val}(f) .\end{aligned}$$

□

Theorem 13.4. *For a flow f and any cut $[S]$ of N,* $\mathrm{val}(f) \le \alpha[S]$. *Furthermore, equality holds if and only if for each $u \in S$ and $v \notin S$,*

(i) *if $e = uv \in E_N$, then $f(e) = \alpha(e)$,*
(ii) *if $e = vu \in E_N$, then $f(e) = 0$.*

Proof. By the definition of a flow,

$$f^+(S) = \sum_{e \in [S]} f(e) \le \sum_{e \in [S]} \alpha(e) = \alpha[S] ,$$

and $f^-(S) \ge 0$. By Lemma 13.3, $\mathrm{val}(f) = \mathrm{val}(f_S) = f^+(S) - f^-(S)$, and hence $\mathrm{val}(f) \le \alpha[S]$, as required.

The equality holds if and only if (1) $f^+(S) = \alpha[S]$ and (2) $f^-(S) = 0$. Here (1) holds if and only if $f(e) = \alpha(e)$ for all $e \in [S]$ (since $f(e) \le \alpha(e)$), and (2) holds if and only if $f(e) = 0$ for all $e = vu$ with $u \in S$, $v \notin S$. This was the claim. □

In particular, if f is a maximum flow and $[S]$ a minimum cut, then

$$\mathrm{val}(f) \le \alpha[S] .$$

Corollary 13.5. *If f is a flow and $[S]$ a cut such that* $\mathrm{val}(f) = \alpha[S]$, *then f is a maximum flow and $[S]$ a minimum cut.*

Ford and Fulkerson proved in 1955 the main result of network flows.

Theorem 13.6. *A flow f of a network N is maximum if and only if there are no f-improvable paths in N.*

Proof. By Lemma 13.2, a maximum flow cannot have improvable paths.

Conversely, assume that N contains no f-improvable paths, and let

$$S_I = \{u \in V_N \mid \text{exists a path } P\colon s \xrightarrow{\star} u \text{ with } \iota(P) > 0\}\,.$$

Set $S = S_I \cup \{s\}$.

Consider an edge $e = uv \in E_N$, where $u \in S$ and $v \notin S$. Since $u \in S$, there exists a path $P\colon s \xrightarrow{\star} u$ with $\iota(P) > 0$. Moreover, since $v \notin S$, $\iota(Pe) = 0$ for the path $Pe\colon s \xrightarrow{\star} v$. Therefore $\iota(e) = 0$, and so $f(e) = \alpha(e)$.

By the same argument, for an edge $e = vu \in E_N$ with $v \notin S$ and $u \in S$, $f(e) = 0$.

By Theorem 13.4, we have $\text{val}(f) = \alpha[S]$. Corollary 13.5 implies now that f is a maximum flow (and $[S]$ is a minimum cut). □

Theorem 13.7. *Let N be a network, where the capacity function $\alpha\colon V \times V \to \mathbb{N}$ has integer values. Then N has a maximum flow with integer values.*

Proof. Let f_0 be the zero flow, $f_0(e) = 0$ for all $e \in V \times V$. A maximum flow is constructed using Lemma 13.2 by increasing and decreasing the values of the edges by integers only. □

The proof of Theorem 13.6 showed also

Corollary 13.8 (Max-Flow Min-Cut)**.** *In a network N, the value* $\text{val}(f)$ *of a maximum flow equals the capacity $\alpha[S]$ of a minimum cut.*

Applications to graphs*

The Max-Flow Min-Cut Theorem is a strong result, and many of our previous results follow from it.

Example 13.9. We mention a connection to the Marriage Theorem, Theorem 5.3. For this, let G be a bipartite graph with a bipartition (X, Y), and consider a network N with vertices $\{s, r\} \cup X \cup Y$. Let the edges (with their capacities) be $sx \in E_N$ ($\alpha(sx) = 1$), $yr \in E_N$ ($\alpha(yr) = 1$) for all $x \in X$, $y \in Y$ together with the edges $xy \in E_N$ ($\alpha(xy) = |X| + 1$), if $xy \in E_G$ for $x \in X$, $y \in Y$. Then G has a matching that saturates X if and only if N has a maximum flow of value $|X|$. Now Corollary 13.8 gives Theorem 5.3. □

Next we apply the theorem to **unit networks**, where the capacities of the edges are equal to one ($\alpha(e) = 1$ for all $e \in E_N$). We obtain results for (directed) graphs.

Lemma 13.10. *Let N be a unit network with source s and sink r.*

(i) *The value* $\text{val}(f)$ *of a maximum flow equals the maximum number of edge-disjoint directed paths $s \xrightarrow{\star} r$.*

(ii) *The capacity of a minimum cut $[S]$ equals the minimum number of edges whose removal destroys the directed connections $s \xrightarrow{\star} r$ from s to r.*

Proof. Omitted. □

Therefore

Corollary 13.11. *Let u and v be two vertices of a digraph D. The maximum number of edge-disjoint directed paths $u \xrightarrow{\star} v$ equals the minimum number of edges, whose removal destroys all the directed connections $u \xrightarrow{\star} v$ from D.*

Corollary 13.12. *Let u and v be two vertices of a graph G. The maximum number of edge-disjoint paths $u \xrightarrow{\star} v$ equals the minimum number of edges, whose removal destroys all the connections $u \xrightarrow{\star} v$ from D.*

The next corollary is **Menger's theorem**.

Corollary 13.13. *A graph G is k-edge connected if and only if any two distinct vertices of G are connected by at least k independent paths.*

Seymour's 6-flows*

DEFINITION. A **k-flow** (H, α) of an undirected graph G is an orientation H of G together with an edge colouring $\alpha\colon E_H \to [1, k-1]$ such that for all vertices $v \in V$,

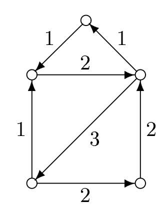
$$\sum_{e=vu\in E_H} \alpha(e) = \sum_{f=uv\in E_H} \alpha(f)\,, \tag{13.1}$$

that is, the sum of the incoming values is equal to the sum of the outgoing values. □

Note that in the k-flows we do not have any source or sink. For convenience, let $\alpha(e^{-1}) = -\alpha(e)$ for all $e \in E_H$ in the orientation H of G so that the condition (13.1) becomes

$$\sum_{e=vu\in E_H} \alpha(e) = 0\,. \tag{13.2}$$

A graph with a 4-flow:



The condition (13.2) generalizes to the subsets $A \subseteq V_G$ in a natural way,

$$\sum_{\substack{v\in A, u\notin A\\ e=vu}} \alpha(e) = 0\,, \tag{13.3}$$

since the values of the edges inside A cancel out each other. In particular,

Lemma 13.14. *If G has a k-flow for some k, then G has no bridges.*

Research problem. It was conjectured by TUTTE (1954) that *every bridgeless graph has a* 5*-flow.* The Petersen graph has a 5-flow but does not have any 4-flows, and so 5 is the best one can think of. Tutte's conjecture resembles the 4-Colour Theorem, and indeed, the conjecture is known to hold for the planar graphs. The proof of this uses the 4-Colour Theorem.

In order to fully appreciate Seymour's result, Theorem 13.15, we mention that it was proved as late as 1976 (by JAEGER) that every bridgeless G has a k-flow for *some* integer k.

SEYMOUR's (1981) remarkable result reads as follows:

Theorem 13.15. *Every bridgeless graph has a* 6*-flow.*

Proof. Omitted. □

DEFINITION. The **flow number** $f(G)$ of a bridgeless graph G is the least integer k for which G has a k-flow. □

Theorem 13.16. *A connected graph G has a flow number $f(G) = 2$ if and only if it is Eulerian.*

Proof. Exercise. □

Theorem 13.17. *For a clique K_n,*

$$f(K_n) = \begin{cases} 2 & \textit{if } n \textit{ is odd}, \\ 3 & \textit{if } n \textit{ is even}. \end{cases}$$

Proof. Exercise. □