ON THE FUNCTION WHICH IS DEFINED BY THE MINIMAL SOLUTION TO $n^2 + f^2(n) = a^2 + b^2$, $a, b, f(n) > n$

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Abstract. For every $n \in \mathbb{N}$, let $f(n) > n$ be the smallest integer for which there exists two integers $a, b > n$ satisfying $n^2 + f^2(n) = a^2 + b^2$. This is the sequence A360796 in OEIS. The aim of this paper is to investigate values of n such that $f(n) = f(n+1)$.

1. Introduction

We start with the definition of the function in question.

Definition 1. *For every positive integer n, let $f(n) > n$ be the smallest integer for which there exist two integers $a > n$ and $b > n$ such that*

$$n^2 + f(n)^2 = a^2 + b^2.$$

The sequence $\{f(n) : n \in \mathbb{N}\}$ is well-defined due to the identity $n^2 + (2n+5)^2 = (n+4)^2 + (2n+3)^2$, showing that $f(n) \leq 2n+5$. Actually, the largest n when this bound is achieved is $n = 16$. The sequence is labelled A360796 in [**?**] with first 50 terms being as follows (the offset is $n = 1$):

$$7, 9, 11, 13, 14, \mathbf{17}, \mathbf{17}, 19, 20, 25, 23, 29, 26, 27, 29, 37, 31, 40, 34, 35, 38, 46, 39, 41, 44,$$
$$43, 44, 54, 47, 58, 49, 51, 56, 53, 54, 67, 62, \mathbf{59}, \mathbf{59}, 70, 62, 73, 64, 65, 74, 78, 69, \mathbf{71}, \mathbf{71}, 75.$$

Consecutive equal values are marked in bold. There appears to be infinitely such pairs, and this phenomenon constitutes the main topic of our paper.

Let us therefore define two ordered sets

$$\mathscr{A} = \Big\{[n, f(n)] : n \in \mathbb{N}, f(n) = f(n+1)\Big\}, \quad \mathscr{T} = \Big\{n \in \mathbb{N} : f(n) = f(n+1)\Big\}.$$

The set $\mathscr{T}$ is just a projection of $\mathscr{A}$ onto its first component. Here is the set $\mathscr{T} \cap [1, 1400]$:

6, 38, 48, 63, 94, 131, 142, 160, 174, 207, 223, 278, 284, 339, 362, 373, 390, 406, 474, 493, 587, 643, 644, 712, 758, 798, 807, 814, 831, 849, 987, 998, 1006, 1043, 1158, 1159, 1217, 1246, 1279, 1332, 1369.

It is impossible to fully describe the set $\mathscr{T}$. Indeed, we will soon demonstrate that it can be decomposed into three disjoint subsets

$$\mathscr{T} = \mathscr{T}^1_R \bigcup \mathscr{T}^1_S \bigcup \mathscr{T}^{>1}$$

(notation will be explained soon). The structure of the two subsets (apparently, infinite) heavily depend on the irregularities in the distribution of primes. Consequently, not much (still something) can be said about these two sets.

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n	$f(n)$	(q,w)	n	$f(n)$	(q,w)
807	926	$(1,2)$	1019814	1023863	$(2,1)$
1369	1528	$(1,2)$	1019932	1024001	$(2,1)$
9302	9789	$(1,3)$	1037337	1041416	$(1,2)$
58003	58994	$(1,2)$	1211041	1215448	$(1,2)$
64897	65924	$(1,2)$	1291147	1295702	$(1,2)$
84564	85997	$(1,3)$	1438690	1446389	$(1,5)$
256483	258514	$(1,2)$	1530897	1535852	$(1,2)$
309720	312935	$(2,2)$	1551003	1555994	$(1,2)$
357870	361265	$(2,2)$	1600989	1606058	$(1,2)$
373974	376463	$(2,1)$	1846032	1852769	$(1,3)$
477799	480590	$(1,2)$	1905667	1911194	$(1,2)$
550059	554270	$(2,2)$	1942813	1948534	$(1,2)$
618469	621620	$(1,2)$	2428696	2435173	$(2,1)$
655254	660065	$(2,2)$	2698206	2704799	$(2,1)$
881457	885224	$(1,2)$	2977984	2984893	$(2,1)$
964672	968669	$(2,1)$	3133951	3141098	$(1,2)$

TABLE 1. Solutions to $f(n)=f(n+1)$ and $q(n+1)\cdot w(n+1)>1$

On the other hand, the first component can be described precisely. Let us define

$$P(u,v)=\frac{(uv+u+v-1)(uv-2)}{2}-1,\quad R(u,v)=\frac{uv(u+1)(v+1)}{2}-1.$$

Definition 2. *Consider the region* $\mathbf{T}=\{(u,v)\in\mathbb{Z}^2 : u\geq v\geq 1\}$.

Theorem 1. *Let* $(u,v)\in\mathbb{Z}^2$, $u=v$, *or* $v+2\leq u\leq 6v-3$. *Then*

$$P(u,v)\in\mathscr{T}^1_R.$$

Next, not all pairs $[P(u,v),Q(u,v)]$ belong to $\mathscr{A}$, but the number of exceptions is small.

1.1. **Exceptional case.** If $u=v+1$, then

$$\begin{aligned} P(v+1,v)+1 &= \frac{(v-1)v(v+2)(v+3)}{2}=\frac{(v^2+v-2)(v^2+3v)}{2},\\ P(v+1,v)+2 &= \frac{(v^2+2v-1)(v^2+2v-2)}{2}. \end{aligned}$$

The fact that in this particular case $P(v+1,v)+1$ factors into linear components is responsible for that fact that $f(P(v+1,v))\neq f(P(v+1,v)+1)$. Indeed,

1.2. **Consecutive numbers in $\mathscr{T}$.** We note that $643,644\in\mathscr{T}$. This implies $f(643)=f(644)=f(645)=719$. With the help of our main theorem, in general, we are able to prove the infinitude of n satisfying $f(n)=f(n+1)=f(n+2)$. For this purpose it is enopugh to note that, first

$$P(3v,2v+1)=P(6v+2,v)+1,\quad P(3v+2,2v)=P(6v+3,v)+1,$$

and, second $(3v,2v+1)$, $(6v+2,v)$, $(3v+2,2v)$, $(6v+3,v)$ belong to $\mathscr{H}$.

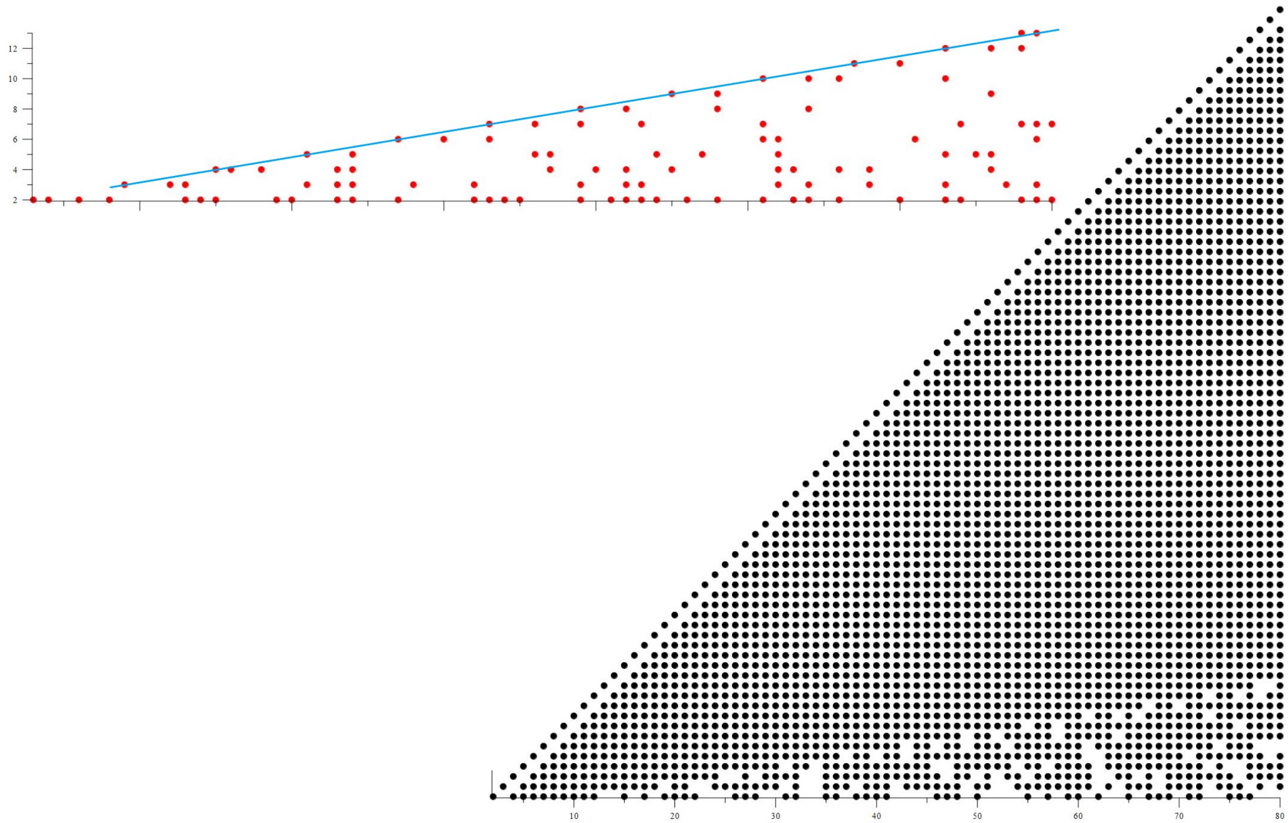

FIGURE 1. The set $\mathscr{G}$, $u \leq 80$ (rights), and the set $\mathscr{H}$ (left, the line $v = u - 1$ is not shown).

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Theorem 2. *Let*

$$\begin{aligned} n &= (6v-1)(v+1)(3v^2+3v-1) \quad v \geq 5, \text{ or} \\ n &= (3v^2+5v+1)(6v^2+3v-2), \quad v \geq 3. \end{aligned}$$

Then $f(n-1) = f(n) = f(n+1)$.

It is known that every prime number $p \equiv 1 \pmod 4$ can be given a unique representation $p = x^2 + y^2$, $x > y$, $x, y \in \mathbb{N}$. It seems that the full description of the set of integers satisfying the condition $f(n) = f(n+1)$ relies on fine arithmetic of these pairs (x, y) (for instance, the infinitude of representations with $y = 1$ has not yet been proved). However, there are elementary ways to provide an infinite sequences satisfying $f(n) = f(n+1)$. In the proof given below, no number theory beyond the notion of greatest common divisor is being used, and also no algebra beyond the formulas for $a^2 - b^2$ and $(a+b)^2$. In the very end of the solution (see Lemma) we need to check a simple fact about factorization of $2n^2 - 1$. It is indeed surprising that no sophistication beyond that is needed.

Solution. To show that $f(n)$ is well-defined, note that $n^2 + (2n+5)^2 = (n+4)^2 + (2n+3)^2$. This demonstrates that $f(n) \leq 2n + 5$. [1]

Proposition 1. *For every* $n \in \mathbb{N}$, *one has* $f(2n^2 - 1) = 2(n+1)^2 - 1$.

[1] Remark for PSC: the largest n when this bound is achieved is $n = 16$.

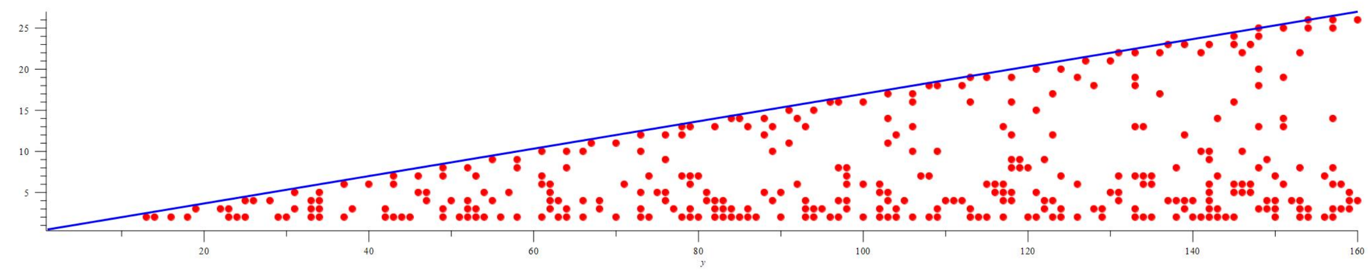

FIGURE 2. ...

Proof. Indeed, the identity

$$(2n^2-1)^2+\big(2(n+1)^2-1\big)^2=(2n^2+2n+1)^2+(2n^2+2n+1)^2 \tag{1}$$

shows that $f(2n^2-1)\leq 2(n+1)^2-1$. Let $N=2n^2-1$. Suppose there exists $c\leq 2n^2+4n$ and $a,b\geq 2n^2$, satisfying

$$N^2+c^2=a^2+b^2,\quad a,b\in[N+1,c-1]. \tag{2}$$

Since $M^2\equiv 1 \pmod 4$ for odd M, and $M^2\equiv 0 \pmod 4$ for even M, parity of one of a,b matches the parity as N, while the parity of the other matches that of c. Let $a\equiv N \pmod 2$. Identity (2) now rewrites as

$$\frac{a-N}{c-b}=\frac{c+b}{a+N}. \tag{3}$$

Let $2D=\text{g.c.d.}(c+b,a+N)$. Note that if $a\leq b$, then $a\leq 2n^2+2n$, otherwise the inequality on c would be breached. Since a is odd, $a\leq 2n^2+2n-1$. If $a>b$, then $a\leq 2n^2+4n-1$. Let $\text{g.c.d.}(a-N,c-b)=2w$. The identity (3) shows that

$$4n^2+2\leq c+b=2D\frac{(a-N)}{2w}\leq\begin{cases}\frac{D}{w}\cdot 4n \text{ (in case } a>b) \Rightarrow D>wn,\\ \frac{D}{w}\cdot 2n \text{ (in case } a\leq b) \Rightarrow D>2wn.\end{cases}$$

Now, note that

$$0 < c + b - a - N \le \begin{cases} c - N - 1 \le 4n \text{ (in case } a > b), \\ 2(c - N) - 2 \le 8n \text{ (in case } a \le b). \end{cases}$$

This shows that $w \ge 2$ cannot occur, since the fact $2D|(c+b-a-N)$ would lead to a contradiction. Hence $w = 1$. This also shows that $c + b - a - N = 2D$, since $c + b - a - N = 4D$ would also lead to a contradiction. So, if $c - b = 2r$, we must have

$$a - N = 2r + 2, \quad c - b = 2r, \quad c + b = (2r + 2)D, \quad a + N = 2rD.$$

Accordingly,

$$\boxed{N = rD - r - 1, \quad a = rD + r + 1, \quad b = rD + D - r, \quad c = rD + D + r.} \tag{4}$$

Now comes a simple though a crucial trick. First, we see that $c - N = D + 2r + 1$. On the the other hand, the AM-GM inequality $(x + y)^2 \ge 4xy$ implies

$$r(D-1) = N + 1 = 2n^2 \Rightarrow 2r(D-1) = 4n^2 \Rightarrow D + 2r - 1 \ge 4n \Rightarrow c \ge 2n^2 + 4n + 1,$$

which is a contradiction. □

Proposition 2. *For every $n \in \mathbb{N}$, one has $f(2n^2 - 2) \ge 2(n+1)^2 - 1$.*

Proof. Let us very carefully repeat the previous series of arguments in case $N = 2n^2 - 2$, starting from the equation (2). Suppose, there exists $c \le 2n^2 + 4n$ and $a, b \ge 2n^2 - 1$, satisfying (2). In the same vein, if $a \le b$, $a \le 2n^2 + 2n$. In fact, the equality cannot be achieved, since then

$$(2n^2 - 2)^2 + c^2 \ge (2n^2 + 2n)^2 + (2n^2 + 2n)^2 \Rightarrow c > 2n^2 + 4n - 1.$$

It is checked directly that $c = 2n^2 + 4n$, $a = b = 2n^2 + 2n$ is not a solution. In short, we show in the same manne that in case $N = 2n^2 - 2$ representation (2) must hold, too. However, full carefulness is needed, since (as an instance) it is untrue that $f(2n^2 - 3) \ge 2(n+1)^2 - 1$ holds (see **Remark 4** below). Apart from representation (4), in case $N = 2n^2 - 2$ the following simple lemma is crucial for the bound to work. □

Lemma. *Let $r, D, n \in \mathbb{N}$, $n \ge 2$. Assume $r(D - 1) = 2n^2 - 1$. Then $D + 2r - 1 \ge 4n + 1$.*

Proof. Indeed, $2r(D - 1) = 4n^2 - 2$, so AM-GM inequality gives $2r + D - 1 > 4n - 1$. However, if $2r + D - 1 = 4n$, then $D - 1 = 2n + t$, $2r = 2n - t$, $t \in \mathbb{Z}$. These two identities cannot hold, since this would imply $2r(D - 1) = 4n^2 - t^2 \ne 4n^2 - 2$. [2] Thus, $2r + D - 1 \ge 4n + 1$.

To finish the proof of Proposition (2), note that Lemma implies $c = N + D + 2r + 1 \ge 2n^2 + 4n + 1$, and this is a contradiction. □

Now we will find when the equality in Proposition 2 holds. Let

$$(2n^2 - 2)^2 + \left(2(n+1)^2 - 1\right)^2 = a^2 + b^2. \tag{5}$$

Suppose $a = 2n^2 + 2n + 1 - s$, $b = 2n^2 + 2n + 1 + t$, $s, t \ge 0$. Numbers s, t must have different parities. The equation simplifies to

$$4n^2 - 3 + s^2 + t^2 = 2(2n^2 + 2n + 1)(s - t).$$

[2] This argument fails for $N = 2n^2 - 3$, since $4n^2 - t^2 = 4n^2 - 4$ does, obviously, have a solution. That is why $f(2n^2 - 3)$ can be given an algebraic formula; namely, it equals $2(n+1)^2 - 3$ for $n \ge 3$.

We additionally have bounds $t \leq 2n-1$, $s \leq 2n+2$. This implies $s-t=1$ or $s-t=2$. The latter, however, is impossible due to parity restriction. And so,

$$s^2+(s-1)^2 = 4n+5 \Rightarrow n = \frac{s(s-1)}{2}-1, \quad s \geq 3.$$

Remarks for Problem Selection Committee

Proposition 3. *Let x, y, z, w be solutions to*

$$\begin{cases} x+y-z-w=1, \\ xy = zw. \end{cases}$$

satisfying $|x-y| <$, $|z-w| <$, Then The formulas (4) for $(r_1, D_1) = \big(\frac{x}{2}-1, y-1\big)$ and, respectively, for $(r_2, D_2) = \big(\frac{z}{2}-1, w-1\big)$, provide a solution to $f(n) = f(n+1)$. This common value is equal to $\frac{xy}{2}-1$.

As one of x, y must be even, as well as one of z, w, let it be the first one, Define a quadruple (N_1, c_1, a_1, b_1) by formulas (4) using a pair (r_1, D_1). Analogously, define a a quadruple (N_2, c_2, a_2, b_2) by formulas (4) using a pair(r_2, D_2). We have

$$\begin{aligned} N_2 - N_1 &= r_2(D_2-1) - r_1(D_1-1) = \Big(\frac{z}{2}-1\Big)\cdot(w-2) - \Big(\frac{x}{2}-1\Big)\cdot(y-2) \\ &= \frac{1}{2}(zw-xy) - z - w + x + y = 1, \\ c_2 - c_1 &= (r_1+1)(D_1+1) - (r_1+1)(D_1+1) = \frac{z}{2}\cdot w - \frac{x}{2}\cdot y = 0. \end{aligned}$$

The solution to equations in Poposition are given by

$$\begin{cases} x = uv, \quad y = (u+1)(v+1), \\ z = u(v+1), \quad w = (u+1)v. \end{cases}$$

Then

$$N_2 = \frac{1}{2}(uv+u+v-1)(uv-2).$$

2. General solution

$$\boxed{N = RD - Rw - qw, \quad a = RD + Rw + qw, \quad b = RD + Dq - Rw, \quad c = RD + Dq + Rw.}$$

Let the ordered set of divisors of $N+qw$ be $\{d_1, d_2, \ldots d_L\}$, where $L = \sigma_0(N+qw)$, $d_1 = 1$, $d_L = N+qw$. If $R = d_i$, then $D - w = d_{L+1-i}$. Thus,

$$c - N = 2Rw + Dq + qw = 2d_i w + d_{L+1-i} q + 2qw.$$

Regarding this, let us define

$$f(n; q, w) = \min_{R,D \in \mathbb{N}} (RD + Dq + Rw), \text{ subject to the restriction } RD - Rw - qw = n.$$

The following MAPLE procedure calculates the value.

```
a:=proc(n,q,w)::integer;
local S,L,M,i;
S:=Divisors(n+q*w):
L:=nops(S):
M:=n*q+w*(q^2+2*q+2):
for i from 1 to L do
M:=min(M,2*S[i]*w+S[L+1-i]*q+2*q*w):
end do:
n+M:
end proc:
```

In this notation,

$$f(n) = \min_{q,w\in\mathbb{N}} f(n;q,w).$$

Proposition 4. *If $q(n) \cdot w(n) > 1$, then $f(n+1) < f(n)$.*

Proof. Indeed, assume $q(n)\cdot q(n) > 1$, and let $n+qw = R(D-w)$. Then $n+1+qw-1 = R(D-w)$, □

We see that any quadruple $(N, c; a, b)$ comes from specific factorization of $N + qw = R(D - w)$. We can interchange the roles of q, w, as well as of those to factors. This gives

$$N(R, D; q, w) = N(D - w, R + w; q, w) = N(R, D - w + q; w, q) = N(D - w, R + q; w, q).$$

Proposition 5. *We have $f(n; q, w) = f(n; w, q)$.*

Proof. Indeed, this follows from the identity

$$N + qw = R(D - w) = (D - w)(R + q - q) = R_0(D_0 - q), \text{where } R_0 =$$

$$c(R_0, D_0; w, q) = c(D - w, R + q, w, q)$$

□

We will rewrite this equation in three particular case $(q, w) = (1, 1)$, $(q, w) = (1, 2)$ and $(q, w) = (2, 1)$.

Suppose $N+2 = xy$. The second formula gives $N-c = 4x+y+4$. The third one $N-c = 2x+2y+6$.

Let q be prime such that $p = 2q - 1$ is prime, too. Put $N = p - 1$. The minimal value for c given by these formulas (for $q > 10$) occurs for $(q, w; R, D) = (1, 2; 2, q + 2)$. namely, it is $c = 3q + 10$.

Calculation show that under these assumptions, $f(n) = 3n+10$ indeed holds for $n = 19, 31, 37$, but fail for further examples $q = 79, 97, 139, \ldots$ (it is not yet known that the number of primes of the form $2q-1$ is infinite). Let us take $q = 79$. Then the true quadruple $(2q-2, c; a, b)$ is $(156, 233; 180, 215)$. It is achieved for $(q, w; R, D) = (1, 2; 3, 56)$. This is a particularly good example, since $N + 3 = 3r$, where r is also prime.

As a final remark leadinhg to a general understanding of complication related to this problem, consider integers N such that $N + 1 = p$, $N + 2 = 2q$, $N + 3 = 3q$, where (p, q, r) is a triple of primes.

$$N = 6k = p - 1 = 2q - 2 = 3q - 3.$$

A278583

3. Solutions to $f(N)=f(N+1)$ arising from factorizations of $N+1$ and $N+3$

We have seen that equation (**??**) has two solution $N=807$ and $N=1369$ which are nor covered by the construction of the previous chapter. Indeed, in both cases the first of the quadruples $(N,c;a,b)$ arises from the factorization of $N+1$ ($(q,w)=(1,1)$), while the second quadruple $(N+1,c,\hat{a},\hat{b})$ arises from factorization of $N+3$ ($(q,w)=w,1$). Thus, let Let $N+1=xy$, $f(N)=xy+2x+y+1$, $N+3=zw$, $f(N+1)=zw+2z+2w+2$. Thus, we need to find natural positive integer solutions to

$$\begin{cases} 2x+y=2z+2w+3, \\ xy+2=zw, \\ \text{if } xy+1=uv \Rightarrow 2u+v+1>2x+y. \end{cases} \tag{6}$$

Quadruples $(x,y;z,w)=(8,101;30,27)$, $(10,137;28,49)$, $(68,853;194,299)$ or $(74,877,236,275)$ give rise to solutions of (**??**) in case $N=808$, and, respectively, $N=1370$. The first equation shows that y is odd. There are two ways to look at the system (8).

3.1. Quadratic imaginary field.

Multiply the first equation of (8) by α, the second one by a factor 2, and add. We wish to obtain the identity

$$(2x+\alpha)(y+\alpha)=2(z+\alpha)(w+\alpha). \tag{7}$$

In order this to be the case, α should satisfy

$$\alpha^2-3\alpha+4=0.$$

And so, we will be working in the field $\mathbf{K}=\mathbb{Q}(\sqrt{-7})$ of class number 1. Let $\pi=\frac{1+\sqrt{-7}}{2}$, $\varrho=\frac{1-\sqrt{-7}}{2}$. In particular, we can take $\alpha=-\pi^2=\frac{3-\sqrt{-7}}{2}$. Its conjugate is $\beta=-\varrho^2=\frac{3+\sqrt{-7}}{2}$. Note that $\pi\varrho=2$. Taking the norm of (7) gives

$$(4x^2+6x+4)(y^2+3y+4)=4(z^2+3z+4)(w^2+3w+4).$$

In case of the quadruples $(x,y;z,w)=(8,101;30,27)$ and $(x,y;z,w)=(10,137;28,49)$ this reduces to factorization, respectively,

$$\begin{aligned} (2^2\cdot 7\cdot 11)\times(2^2\cdot 37\cdot 71) &= 2^2\times(2\cdot 7\cdot 71)\times(2\cdot 11\cdot 37), \\ (2^4\cdot 29)\times(2^4\cdot 11\cdot 109) &= 2^2\times(2^3\cdot 109)\times(2^3\cdot 11\cdot 29) \end{aligned}$$

Primes $\equiv 1,2,4 (\mathrm{mod}\ 7)$ split in $\mathbb{K}$ (so, $11,37,71,109$ among them), while 7 ramifies. These two quadruples thus provide solutions to (8) of slightly different quality.

3.2. Two-sheeted hyperboloid.

In the other direction, let us substitute the value of y from the first equation of (8) into the second. We therefore obtain a quadratic equation

$$2x^2-x\cdot(2z+2w+3)+zw-2=0.$$

By a direct calculation,

$$D=(2z+2w+3)^2-8(zw-2)=(2z+3)^2+(2w+3)^2+7.$$

Thus, if $X^2=D$, $2z+3=Z$, $2w+3=W$, we have an integer point on a two-sheeted hyperboloid

$$Z^2+W^2-X^2=-7, \quad x=\frac{Z+W-3\pm X}{4}, \quad y=\frac{Z+W-3\mp X}{2}.$$

(q,w)	N	a	b	c	$c-N$
$(1,1)$	$RD-R-1$	$RD+R+1$	$RD+D-R$	$RD+D+R$	$D+2R+1$
$(1,2)$	$RD-2R-2$	$RD+2R+2$	$RD+D-2R$	$RD+D+2R$	$D+4R+2$
$(2,1)$	$RD-R-2$	$RD+R+2$	$RD+2D-R$	$RD+2D+R$	$2D+2R+2$
$(3,1)$	$RD-R-3$	$RD+R+3$	$RD+3D-R$	$RD+3D+R$	$3D+2R+3$

The sign is chosen such that x is an integer. As we have seen, there is the unique choice. These are all solutions for $1\le w\le z\le 30$:

$$(0,9;\mathbf{2},1),(7,1;3,3),(8,1;5,\mathbf{2}),(2,31;8,8),(1,25;\mathbf{9},3),(2,35;12,6),(18,3;14,4),(22,5;\mathbf{16},7),$$
$$(30,9;17,\mathbf{16}),(2,49;20,5),(33,9;\mathbf{23},13),(7,89;25,25),(32,5;27,6),(8,101;\mathbf{30},27).$$

Two special series of solutions are as follows:

$$m,(m+3)^2;\frac{(m+1)(m+4)}{2},m+1.$$

The bold values of integers M are those that $7|(M^2+3M+4)$. This is equivalent to $M\equiv 2 \pmod 7$. Alternatively, this can be derived from the property $\sqrt{-7}|(M+\alpha)$, since $\alpha+2=\frac{7-\sqrt{-7}}{2}=-\sqrt{-7}\pi$. Thus, $\sqrt{-7}|(M+\alpha)\Rightarrow\sqrt{-7}|\big((M-2)+(\alpha+2)\big)\Rightarrow\sqrt{-7}(M-2)\Rightarrow 7|(M-2)$.

4. Solutions to $f(N)=f(N+1)$ arising from factorizations of $N+1$ and $N+4$

4.1. **Our problem.** In relation to our problem, we need to investigate an extension of (8). namely,

$$\begin{cases} 2x+y=2z+3w+5,\\ xy+3=zw. \end{cases} \tag{8}$$

$(x,y;z,w)=(21,443;99,94)$

4.2. **Quadratic field.** In the same manner, mulitply the first equation by γ, add to the first. We wish to obtain

$$(x+\gamma)(y+2\gamma)=(z+3\gamma)(w+2\gamma).$$

Thus,

$$4\gamma^2=5\gamma-3\Rightarrow\gamma=\frac{5+\sqrt{-23}}{8}.$$

$$(4x+4\gamma)(2y+4\gamma)=(4z+12\gamma)(2w+4\gamma).$$

Taking the norm, we get

$$(4x^2+5x+3)(2y^2+5y+6)=(4z^2+15z+27)(2w^2+5w+6).$$

4.3. **Two-sheeted hyperboloid.**

$$x(2z+3w+5-2x)+3=zw,$$
$$2x^2-x(2z+3w+5)+zw-3=0.$$

$$D=(2z+3w+5)^2-8(zw-3)=T^2.$$

5. ..

in this section we we solve the equationb

$$P(u,v)=P(x,y)+1\Rightarrow(uv+u+v-1)(uv-2)=(xy+x-2)(xy+y-2).$$

Example $6\cdot 6=4\cdot 9$ gives the first example. **Remark 1.** We have seen that the equality in case ii) holds if and only if $n=\frac{s(s+1)}{2}-1$, $s\in\mathbb{N}$, $s\geq 2$. For such n if follows that $f(2n^2-2)=f(2n^2-1)$. Expressing explicitly in terms of s, the sequence $\mathcal{K}=\big\{\frac{s(s+1)(s^2+s-4)}{2}:s\geq 2\big\}$ provides a solution to a harder version of the problem. See **Remark 8** where it is shown how a student might arrive at an idea to consider exactly the pair $2n^2-1$ and $2n^2-2$ even if a harder version of the problem is presented. On the other hand, if a simpler version is posed to a student, it is straightforward to demonstrate that equality in part ii) holds if and only if $n=\frac{s(s+1)}{2}-1$. There is no need to prove inequality for this purpose. This is e an advantage of the problem, since 2/7 points might be earned even without fully solving parts i) or ii). Identities (1) and (in case $n=\frac{s(s+1)}{2}-1$) (5) are much easier to obtain.

Remark 2. The first 30 terms of the sequence $f(n)$, $n\in\mathbb{N}$, are as follows:

7, 9, 11, 13, 14, 17, 17, 19, 20, 25, 23, 29, 26, 27, 29, 37, 31, 40, 34, 35, 38, 46, 39, 41, 44, 43, 44, 54, 47, 58.

MAPLE code which takes as in input n and gives as an output the quadruple $(n,f(n),a,b)$ is as follows:

```
a :=proc(n::integer)::List[1..4];
local found::boolean;
local N, SQ, i, c;
found:=false; N:=n+1; SQ:={};
while not found do
SQ:=SQ union {N^2}; N:=N+1;
for i from n+1 to N-1 do
        if evalb(N^2+n^2-i^2 in SQ) then found:=true; c:=i; end if;
                 end do;
            end do;
[n,N,simplify((N^2+n^2-c^2)^(1/2)),c]
end proc:
```

Remark 3. The sequence $f(2n^2-2)-2(n+1)^2+1$, $n\geq 2$, starts from

$\mathbf{0}, 6, 9, \mathbf{0}, 11, 15, 17, \mathbf{0}, 18, 23, 6, 27, \mathbf{0}, 26, 22, 35, 31, 39, \mathbf{0}, 36, 45, 47, 41, 51, 53, \mathbf{0}, 51, 6, 51, 22, 6, 56, \ldots$

Remark 4. Carefulness is needed indeed, since the argument breaks, say, for $N=2n^2-3$. Truly, we have the following result.

Proposition 6. *For $n\geq 3$, one has $f(2n^2-3)=2(n+1)^2-3$, and the corresponding identity is*

$$(2n^2-3)^2+\big(2(n+1)^2-3\big)^2=(2n^2+2n-3)^2+(2n^2+2n+1)^2.$$

If we try similarly to compute $f(2n^2-4)-2(n+1)^2+3$, $n\geq 4$, this would lead to a sequence

$$7, 9, 4, \mathbf{0}, 2, 17, 18, 19, \mathbf{0}, 23, 24, 29, 28, 2, \mathbf{0}, 33, 34, 36, 38, 39, 47, \mathbf{0}, 4, 18, 48, 2, \ldots$$

Value 0 is achieved for $n=\frac{s(s+1)}{2}-3$, $s\geq 4$. Thus, with the same technique as before it is possible to show that the sequence $\mathcal{L}=\big\{\frac{(s^2-6)(s^2+2s-5)}{2}-1:s\geq 4\big\}$ also provides a solution to a harder problem. In the other direction, if we fix M, the sequence $f(2n^2-M)-2(n+1)^2$ (as a rule) behaves unpredictably. The subsequence of M's given by $M = 2s^2 + 1$, $s \in \mathbb{N} \cup \{0\}$, is an exception. Indeed, it can be shown that the following holds. If s is fixed, then

$$f(2n^2 - 2s^2 - 1) = 2(n+1)^2 - 2s^2 - 1 \text{ for sufficiently large } n.$$

Remark 6. With the same technique as used in the proof of Proposition 1, one can show that

$$f(N) \geq N + \sqrt{8(N+1)} + 2,$$

end the equality holds if and only if $N = 2n^2 - 1$.

Remark 7. Which polynomial identities give solutions to $f(n) = f(n+1)$? Suppose $f(n) = f(n+1)$. This implies the existence of positive integers a, b, c, d, e, such that

$$\begin{aligned} n^2 + c^2 &= a^2 + b^2, \\ (n+1)^2 + c^2 &= d^2 + e^2. \end{aligned}$$

Now, if we fix $d - a = t$, this gives the unique identity, as can be demonstrated in a special example $t = 3$. Assume therefore $d = a + 3$. This, after some calculation, one is lead to a pair of identities

$$\begin{aligned} (3n^2 - 2n - 2)^2 + (3n^2 + 3n - 1)^2 &= (3n^2 - 2)^2 + (3n^2 + n + 1)^2, \\ (3n^2 - 2n - 1)^2 + (3n^2 + 3n - 1)^2 &= (3n^2 + 1)^2 + (3n^2 + n - 1)^2. \end{aligned}$$

So, for which n do both identities $f(3n^2 - 2n - 2) = f(3n^2 - 2n - 1) = (3n^2 + 3n - 1)$ hold? It appears to happen quite often. For example, for $200 \leq n \leq 230$, these are the needed values:

$$201, 202, 206, 207, 211, 212, 215, 218, 219, 221, 225, 227, 230.$$

Thsi approach might also lead to a solution of a harder problem.

Remark 8. Even in the case of harder version, one can arrive to the easier one via a logical path. Namely, a general solution to equation in integers $M^2 + c^2 = a^2 + b^2$ can be given several equivalent forms. We will provide one, particularly suited for our problem. Assume (as we have) $a, b \in [M+1, c-1]$. Let parities of M and a match, as well as those of b and c. Thus,

$$\frac{a - M}{c - b} = \frac{c + b}{a + M} > 1.$$

Let g.c.d.$(a - M, c - b) = 2w$. Thus let $a - M = 2Rw + 2qw$, $c - b = 2Rw$, $R, q \in \mathbb{N}$, g.c.d.$(R, q) = 1$. Let, g.c.d.$(c + b, a + M) = 2D$. A standard analysis shows that the solution is given by

$$M = RN - (R+q)w, \quad a = RN + (R+q)w, \quad b = (R+q)N - Rw, \quad c = (R+q)N + Rw,$$

where N is a certain multiple of D. Any choice of $R, N, q, w \in \mathbb{N}$ provides a solution to $M^2 + c^2 = a^2 + b^2$. However, if we want M, a, b, c be large positive integers, R, N should be large as well. On the other hand, in order $c - M = 2Rw + Nq + qw$ to be as small as possible in comparison to M, integers q, w should be as small as possible. Take $q = w = 1$. Then $M = R(N - w) - qw = R(N-1) - 1$. If this value is fixed, we want to minimize the difference $c - M = 2Rw + Nq + qw = 2R + N + 1$. We thus naturally arrive at the condition $2R = N - 1 = 2n$, and this gives $M = 2n^2 - 1$. Thus, even in the harder formulation of the problem it is very natural to try first $f(2n^2 - 1)$.

Remark 9. Let $\big(n, f(n), a, b\big)$ be the quadruple in the formulation of the problem. MAPLE program presented above shows that $a + b - n - f(n)$ attains only the values 2 and 4, the sequence being

$$2 \text{ (seventeen times in a row)}, 4, 2, 2, 2, 4, 2, 2, 2, 2, 2, 4, 2, 4, \ldots$$

Indices at which value 4 occurs are as follows:

$$18, 22, 28, 30, 36, 40, 42, 46, 52, 58, 60, \mathbf{61}, 66, 70, 72, \mathbf{73}, 78, 82, \mathbf{85}, 88, \mathbf{93}, 96, 100, \ldots$$

The non-bold ones are immediately recognized as $p-1$, $p \geq 19$ being a prime number (the bold ones are of the form $2p+1$ for some special primes). On the other hand, $b+c-a-f(n)-2\cdot \text{g.c.d.}\big(b+c, a+f(n)\big)=0$ for all $n \in \mathbb{N}$ without exceptions. Thus, this reasoning yields the following conclusion.

Proposition 7. *Suppose $N \in \mathbb{N}$. If $2(N+1)=KL$ for $K, L \in \mathbb{N}$ being relatively close to one another, then the representation (4) holds.*

We do not specify here the meaning of the phrase "relatively close". In our solution, $n=\frac{s(s+1)}{2}-1$, $N=2n^2-2$. Thus, $2(N+1)=(s^2-2)(s^2+2s-1)$, $2(N+2)=(s+2)^2(s-1)^2$. Yet, a student might find another way to arrive at the representation (4), which always holds for N if $f(N)$ can be given a simple algebraic formula.

$$P=\frac{x(x+1)(x+2)(x+3)}{2}.$$

Then we have identities

$$\begin{aligned}\Big(P-(x+1)(x+2)\Big)^2+\Big(P+(x+1)(x+2)-1\Big)^2 &= \Big(P-(x+1)\Big)^2+\Big(P+(x+2)\Big)^2,\\ \Big(P-(x+1)(x+2)+1\Big)^2+\Big(P+(x+1)(x+2)-1\Big)^2 &= 2\Big(P+1\Big)^2.\end{aligned}$$

Let $q=x^2+3x+1$. Then $(x+1)(x+2)=q+1$, $x(x+3)=q-1$. Thus,

$$P=\frac{(q+1)(q-1)}{2}\Rightarrow q^2=2P+1=(P+1)^2-P^2.$$

$$2(P+1)^2=2(P^2+q^2)=(P-q)^2+(P+q)^2.$$

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