Piercing a Chessboard by Lines from Two Slope Classes

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Abstract

Let an $n \times n$ square board be divided into unit cells, and say that a line *pierces* a cell if it meets its interior. Keller asked in 1980 whether $n-1$ lines are always necessary to pierce all cells; the conjecture is still open. We prove the conjectured lower bound for a nontrivial two-directional subclass. Suppose that every line has slope in $(0, 1)$ or in $(-\infty, -1)$, and that one of the two slope classes contains at most two lines. Then at least $n-1$ lines are necessary, and this is sharp for every $n \geq 3$. The proof passes from lines to four types of line-monotone cell paths and establishes exact obstruction bounds for one and for two ascending-shallow paths. We also record computational examples showing that the unrestricted line-monotone path relaxation is strictly broader than the straight-line problem.

Keywords. chessboard piercing; digital line; monotone path; lattice path; discrete geometry.

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1 Introduction and historical note

Let p_n be the least number of straight lines that pierce all n^2 cells of an $n \times n$ board. The problem appears in the *American Mathematical Monthly* as Keller's Problem E2862 (1980): prove that $n-1$ lines suffice, and decide whether they are necessary [6]. The present author independently formulated the same problem around 1998. A dated 2004 problem note remains in the author's archive [2], and a public formulation, explicitly recorded as a problem from about 1998, appeared in 2016 [3, Problem 9].

The question has since reappeared several times. Ackerman and Pinchasi were motivated by a question raised by Lapid Harel in a Technion course in 2012. They replaced straight lines by ascending and descending staircase walks and proved that the exact minimum in that relaxation is $\lceil 2n/3 \rceil$ [1]. Their paper explicitly says that the line problem remains open and that the authors were unaware of related work. Kerimov extended the staircase-walk result to rectangular boards [7].

A separate development began with a question of Bárány and Frankl [5]. Ambrus, Bárány, Frankl and Varga proved $0.7n < p_n \leq n-1$ for sufficiently large n and formulated the conjecture $p_n = n-1$ [4]. Richter subsequently determined the exact covering number of a rectangle by monotonous polyominoes [8]; for a square his formula again gives $\lceil 2n/3 \rceil$. The bibliographies of these strands of the literature do not fully overlap. Thus the subject has an unusual history of independent rediscoveries, extensions, and re-proofs. A recent survey still records the full equality $p_n = n-1$ as open [9].

The purpose of this paper is a first exact result at the conjectured scale in a genuinely two-directional setting.

Theorem 1.1 (Restricted Keller theorem)**.** *Let $n \geq 3$, and let $\mathcal{L}$ be a finite family of lines piercing every cell of B_n. Assume that every line has slope in*

$$(0,1) \quad \text{or} \quad (-\infty,-1),$$

and that at least one of these two slope classes contains at most two lines. Then $|\mathcal{L}| \geq n-1$. For every $n \geq 3$ equality is attainable.

The proof is entirely elementary. Its main ingredients are a general-position reduction, the digital traces of lines, and two obstruction constructions. The lower bound does not use the normalization lemma of Ackerman–Pinchasi; in particular, paths of the same type are allowed to overlap.

2 From lines to line-monotone paths

We identify B_n with the family of open cells

$$C_{x,y} = (x, x+1) \times (y, y+1), \qquad 0 \leq x, y \leq n-1.$$

A line pierces $C_{x,y}$ if it meets this open square.

Lemma 2.1 (General position)**.** *A finite piercing family may be translated, line by line and parallel to itself, so that no line passes through a lattice point and every cell pierced before the translations is still pierced afterwards.*

Proof. Fix a line ℓ and the finite set of open cells pierced by it. Translate ℓ by a signed distance t in a fixed normal direction. For each of those cells, the set of values of t for which the translated line still meets the cell is an open interval containing 0. Their finite intersection is therefore an open interval containing 0. For each lattice vertex there is at most one forbidden value of t that puts the translated line through that vertex. Choose a nonforbidden t in the common interval. The slope is unchanged. Repeating this independently for all lines proves the lemma. $\square$

For a line in general position, list the pierced cells in the order in which the line meets them from left to right. Consecutive cells share a side. Write $\mathrm{R}, \mathrm{U}, \mathrm{D}$ for a move to the right, up, or down.

Definition 2.2 (Line-monotone paths)**.** A finite rook path is called

Type	allowed steps	forbidden consecutive steps
ascending shallow (AS)	R, U	UU
ascending steep (AT)	R, U	RR
descending shallow (DS)	R, D	DD
descending steep (DT)	R, D	RR

These four classes are called *line-monotone* (LM) paths.

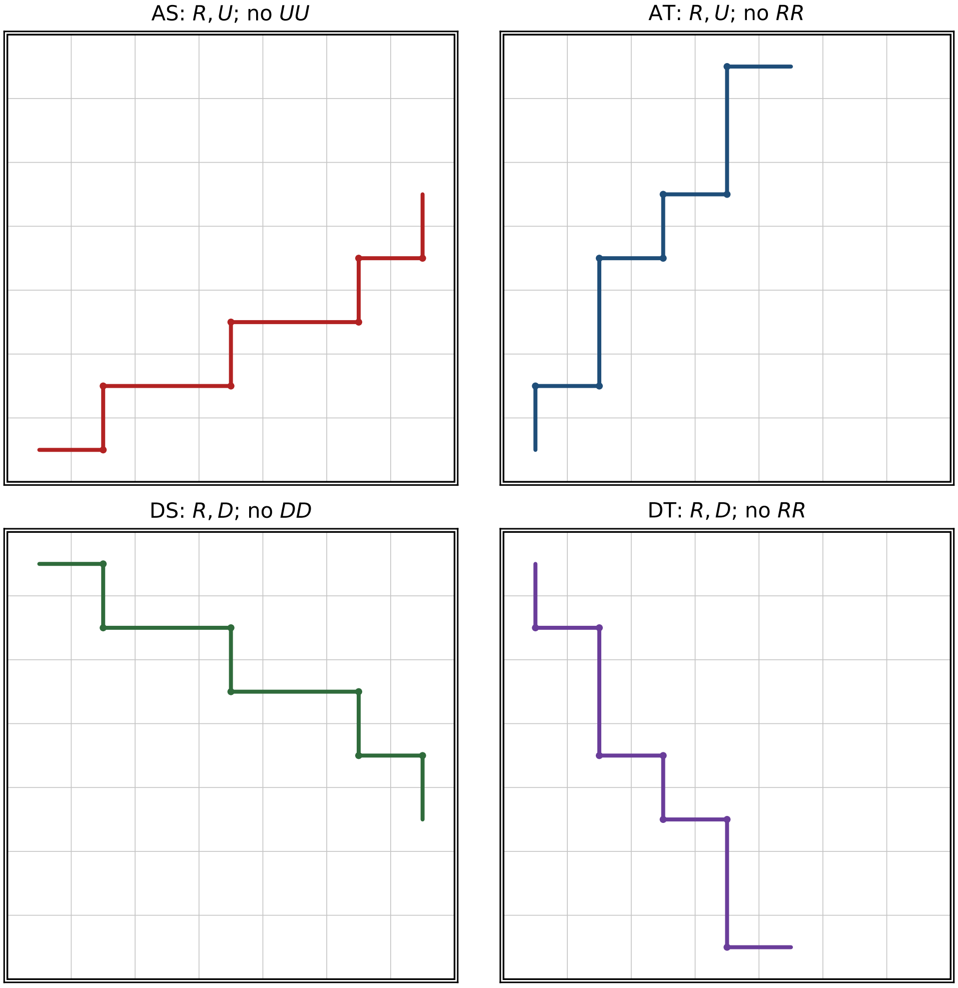


Figure 1: The four LM path types. The present theorem uses AS and DT, but all four types are needed to describe arbitrary straight-line traces.

Lemma 2.3 (Line traces)**.** *After the reduction of Lemma 2.1, a line of slope $m \in (0, 1)$ has an* AS *trace, while a line of slope $m < -1$ has a* DT *trace.*

Proof. For $0 < m < 1$, the trace moves only right or up. Between two consecutive crossings of horizontal grid lines, the horizontal displacement is $1/m > 1$; there is therefore a vertical-grid crossing in between. Hence two upward steps cannot be consecutive.

For $m < -1$, the trace moves only right or down. During a horizontal displacement of one unit, the line drops by more than one unit, and therefore crosses a horizontal grid line. Hence two rightward steps cannot be consecutive. □

A quarter-turn sends a slope m to $-1/m$ and exchanges AS with DT (after reversing the orientation if necessary). Consequently, in proving Theorem 1.1 we may assume that the slope class containing at most two lines is the AS class.

Remark 2.4 (On normalization)*.* Same-type unzipping and disjointification are useful in the general theory of monotone polyominoes. They are not needed below: the one-path argument is valid for every AS path, the two-path argument permits the two AS paths to overlap, and no disjointness is assumed among the DT paths. This removes a potentially delicate normalization step from the proof of Theorem 1.1.

3 Two elementary obstruction principles

We use cell coordinates (x, y), always with $0 \le x, y \le n-1$.

Lemma 3.1. *Let P be a* DT *path.*

(i) If $x_1 < \cdots < x_s$ and $y_1 < \cdots < y_s$, then P contains at most one of the cells (x_i, y_i).
(ii) Suppose the cells (x_i, y_i) have the same checkerboard parity and satisfy

$$x_1 < \cdots < x_s, \qquad x_1 + y_1 < \cdots < x_s + y_s.$$

Then again P contains at most one of them.

Proof. Part (i) follows immediately because the row coordinate of a DT path is nonincreasing as the column coordinate increases.

For (ii), suppose P contains two of the cells in the stated order. Let R and D be the numbers of right and down steps between them. Since the word of a DT path contains no RR,

$$R \le D + 1.$$

On the other hand, the increase of $x + y$ equals $R - D$. It is positive and even, because the two cells have the same parity. Hence $R - D \ge 2$, a contradiction. ☐

4 One ascending-shallow path

Theorem 4.1. *If one* AS *path together with q* DT *paths covers B_n, then $q \ge n-2$. This is sharp for every $n \ge 3$.*

Proof of the lower bound. Let P be the AS path. It contains at most two cells in a column. For a column x met by P, let T_x be its highest occupied row. Whenever both columns x and $x+1$ are met,

$$T_{x+1} \le T_x + 1,$$

so $T_x - x$ is nonincreasing along the columns of P.

If P misses all cells

$$(0, 0), (1, 1), \ldots, (n-3, n-3),$$

these $n-2$ cells form the required obstruction. Otherwise let $a \le n-3$ be the first index with $(a, a) \in P$. Then $T_a \in \{a, a+1\}$. Put

$$r = T_a - a + 1 \in \{1, 2\}$$

and define

$$c_x = \begin{cases} (x, x), & 0 \le x < a, \\ (x, x+r), & a \le x \le n-3. \end{cases}$$

All these cells lie in B_n. The first group is outside P by the minimality of a. For every later column met by P, $T_x \le x + r - 1$, so the second group lies strictly above P; in a column not met by P this is automatic. The column and row coordinates of the cells c_x both increase strictly. By Lemma 3.1(i), every DT path contains at most one of them. Hence $q \ge n-2$. ☐

Sharpness follows from the construction in Section 6 with $k = 1$.

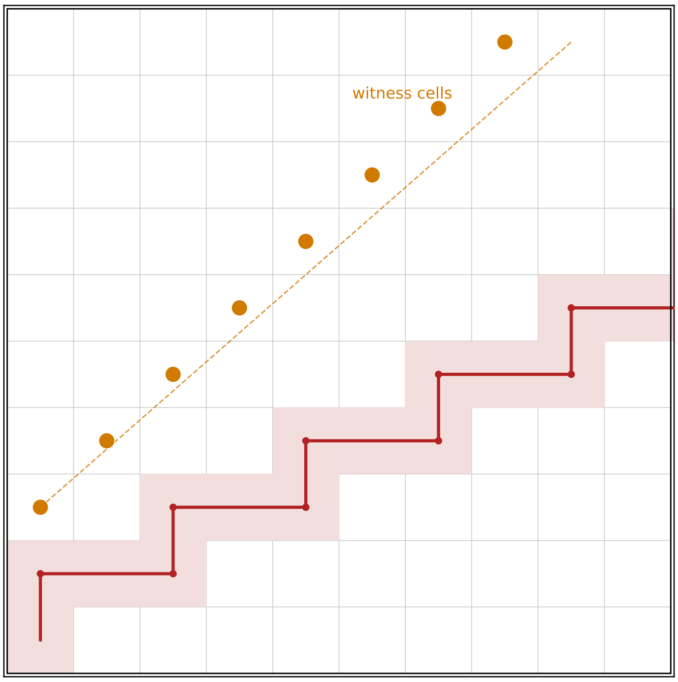


Figure 2: One AS path (red) and an increasing family of $n-2$ witness cells (orange). Every DT path meets at most one witness cell.

5 Two ascending-shallow paths

Theorem 5.1. *If two* AS *paths together with* q DT *paths cover* B_n*, where* $n \geq 4$*, then* $q \geq n-3$*. This is sharp.*

Proof. Let P, Q be the two AS paths; they are not assumed disjoint. For $r = 0, 1, \ldots, n-2$, consider the upper half of the even descending diagonal $x + y = 2r$:

$$E_r^+ = \{(x, 2r - x) : L_r \leq x \leq r\}, \qquad L_r = \max\{0, 2r - n + 1\}.$$

For a cell $(x, 2r - x) \in E_r^+$ define its *lag*

$$\lambda = r - x = \frac{y - x}{2} \geq 0.$$

Between consecutive even-parity cells of an AS path the two-step word is RR, RU, or UR. The lag changes by $-1, 0, 0$, respectively. Thus the lag along an AS even trace is nonincreasing. In particular, once such a trace leaves the upper half, it never returns.

We construct an upper greedy thread. Before processing level r, let c be the column of the last selected cell, with initial value $c = -1$. On E_r^+ choose the cell of smallest column $x > c$ that belongs to neither P nor Q; if no such cell exists, skip the level. The selected cells have one parity, strictly increasing columns, and strictly increasing values of $x + y$. By Lemma 3.1(ii), every DT path meets the thread at most once.

It remains to count skipped levels. Put

$$h = r - c, \qquad m_r = \min\{r, n - 1 - r\}.$$

The candidate lags on level r are precisely

$$0, 1, \ldots, H_r, \qquad H_r = \min\{h - 1, m_r\},$$

and the greedy rule selects the largest available lag. If lag λ is selected, then at the next level $h = \lambda + 1$; after a skip, h increases by one.

Before the first skip one has $h = 1$, so only lag 0 is available. At the first skip some path, say P, occupies lag 0. By lag monotonicity, as long as P remains in the upper half it stays at lag 0. After the skip, $h = 2$ and the candidate lags are $0, 1$.

If P remains present and lag 1 is free, the thread selects lag 1 and keeps $h = 2$. A second skip can occur only when the other path occupies lag 1. If P leaves before this happens, only one path remains capable of blocking the thread. Such a single path can cause at most one further skip: with $h = 2$ it cannot fill both candidate lags; if it first forces the thread to choose lag 0, then it may move from lag 1 to lag 0 and cause one skip, after which lag 1 is permanently free.

Suppose instead that a second skip occurs while both paths are present. The two paths occupy lags 0 and 1, and h becomes 3. Their future lags are at most 0 and 1, so lag 2 can never be occupied by either path. Whenever $m_r \geq 2$, the thread therefore selects lag 2 and keeps $h = 3$. The only later level with $m_r < 2$ is the final level $r = n - 2$, where $m_r = 1$. Thus the upper thread skips at most three levels.

If it skips at most two, it selects at least $(n-1) - 2 = n - 3$ cells, and the proof is complete. Assume that it skips three. The third skip is then on the final level, where the two paths occupy lags 0 and 1. Since lag is nonincreasing forward, every earlier even-parity cell of either path has nonnegative lag: both even traces lie on or above the main diagonal.

Rotate the board through 180° and perform the same upper greedy construction. Under this rotation the lag changes sign. Hence in the rotated upper half the two traces can occupy only lag 0; lag 1 is always free. The rotated thread therefore skips at most once and selects at least $n - 2$ cells. Rotating it back and reversing the order of its selected cells gives a lower obstruction thread in the original board.

In every case the complement of $P \cup Q$ contains at least $n - 3$ cells satisfying the hypotheses of Lemma 3.1(ii). These cells must be covered by distinct DT paths, so $q \geq n - 3$. $\square$

Sharpness follows from Section 6 with $k = 2$.

6 An explicit cross construction

For completeness we give a formula for a cross family realizing every split of $n - 1$ into AS and DT lines. This is an explicit version of the cross pattern discussed in [4].

Proposition 6.1 (Cross construction)**.** *Let $1 \leq k \leq n-2$. The board $[0, n]^2$ can be pierced by k lines of slope $(n-1)/n$ and $n - 1 - k$ lines of slope $-n/(n-1)$. None passes through a lattice point.*

Proof. Put $N = n - 1$ and $H = 2n - 1$. For $0 \leq i \leq k - 1$ take

$$A_i : \quad y = \frac{N}{n}x + \frac{\frac{3}{2} - (k-1)N + iH}{n},$$

and for $0 \leq j \leq N - k - 1$ take

$$D_j : \quad y = -\frac{n}{N}x + \frac{N(k+2) + \frac{3}{2} + jH}{N}.$$

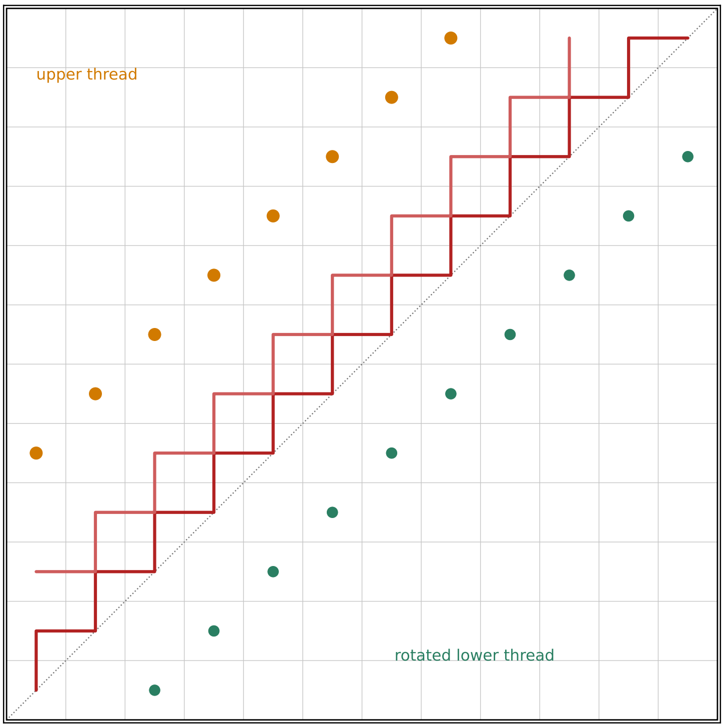


Figure 3: The extreme three-skip situation. The two AS traces occupy lags 0 and 1. The upper thread (orange) loses three levels, but the half-turned lower thread (green) loses at most one.

The half-integer constants show immediately that no line contains a lattice point.

Consider the cell $C_{a,r}$ with lower-left corner (a, r), and set

$$T = nr - Na, \qquad S = Nr + na.$$

A direct comparison of the values of A_i on the two vertical sides of the cell shows that some A_i pierces $C_{a,r}$ exactly when

$$1 - kN \le T \le kn. \tag{6.1}$$

Indeed, the individual intervals of admissible integer values of T are consecutive and are

$$1 - kN + iH \le T \le 1 + (2 - k)N + iH.$$

Similarly, some D_j pierces the cell exactly when

$$kN + 1 \le S \le NH - kn. \tag{6.2}$$

If $T \le -kN$, then $a \ge k$ and $r \le N - k - 1$, which imply (6.2). If $T \ge kn + 1$, then $r \ge k + 1$ and $a \le N - k$, which again imply (6.2). Every cell failing (6.1) is therefore covered by the descending family. The union of the two families pierces the whole board. □

Taking $k = 1$ proves the sharpness assertion in Theorem 1.1 and Theorem 4.1; taking $k = 2$ proves sharpness in Theorem 5.1.

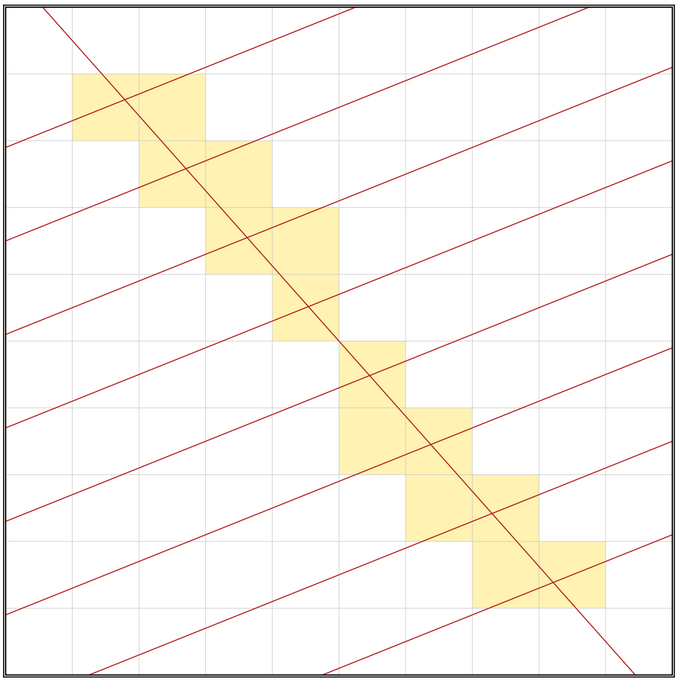


Figure 4: A sharp $n-1$ construction on a 10×10 board: eight parallel lines of slope $2/5$ and one steep transverse line. Pale cells are pierced twice. The construction belongs to the same parallel-plus-transverse family as [4, Theorem 3].

7 Proof of the geometric theorem

Proof of Theorem 1.1. Apply Lemma 2.1 and then Lemma 2.3. After a quarter-turn if necessary, the slope class containing at most two lines gives $p \in \{0, 1, 2\}$ AS paths, while all remaining lines give DT paths.

If $p = 0$, a DT path meets at most one cell of the main diagonal, because $x - y$ increases by one at every step. Hence at least n paths are needed. If $p = 1$, Theorem 4.1 gives at least $n-2$ DT paths, for a total of $n-1$. If $p = 2$ and $n \geq 4$, Theorem 5.1 gives at least $n-3$ DT paths, again for a total of $n-1$. For $n = 3$ the conclusion is immediate once two lines are already present; the remaining cases follow from the preceding alternatives.

Finally, Proposition 6.1 with $k = 1$ gives a piercing family of exactly $n-1$ permitted lines. Thus the bound is sharp. □

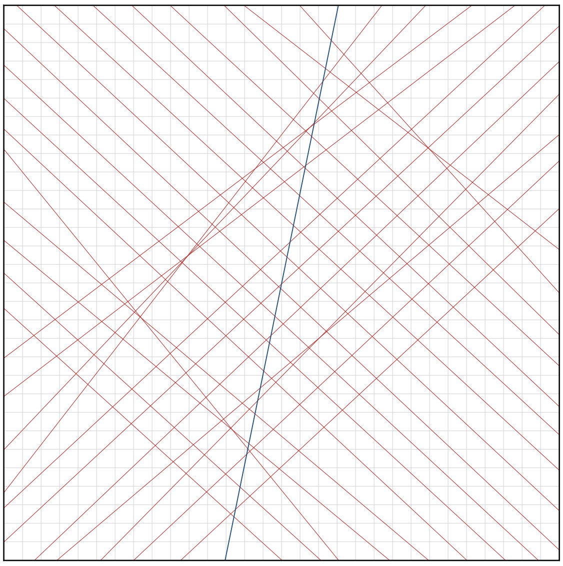
8 Why the LM relaxation is not the final problem

The four LM types isolate important local restrictions of straight-line traces, but arbitrary LM paths retain much more global freedom than digital straight lines. This distinction is real, not cosmetic.

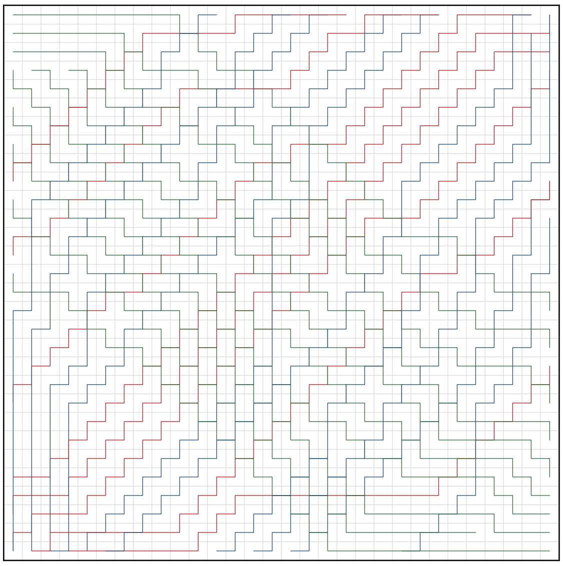
A computer-verified family of $28 = 30 - 2$ LM paths covers the entire 30×30 board. It consists of

$$8\,\mathrm{AS} + 10\,\mathrm{AT} + 10\,\mathrm{DS}$$

paths. Thus the statement "$n-1$ LM paths are always necessary" is false. On the other hand, the same computation led to an irregular full piercing of the 30×30 board by 29 actual lines, with many different slopes; it is neither a two-direction cross family nor a parallel-plus-one



(a) An irregular 29-line full piercing of B_{30}. The final blue line repairs four cells left by the other 28 lines.



(b) A full cover of B_{30} by only 28 LM paths of three types.

Figure 5: Computational landmarks delimiting the scope of the theorem.

family. These two specimens are shown in Fig. 5. Exact path data and line equations are included with the source archive. They are not used in the proof.

The full Keller conjecture therefore requires additional global structure of digital straight lines, beyond the four local LM restrictions. Theorem 1.1 shows, however, that the conjectured value $n-1$ is already forced as soon as one of the two transverse slope classes has size at most two.

Data availability

The source archive accompanying this preprint contains the LaTeX file, programmatic figure sources, exact equations of the irregular 29-line configuration, and the complete cell lists for the 28-path LM cover.

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