TRANSFER OPERATOR FOR THE GAUSS' CONTINUED FRACTION MAP. I. STRUCTURE OF THE EIGENVALUES AND TRACE FORMULAS

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ABSTRACT. Let $\mathcal{L}$ be the transfer operator associated with the Gauss' continued fraction map, known also as the *Gauss-Kuzmin-Wirsing operator*, acting on a Banach space. In this work we prove an asymptotic formula for the eigenvalues of $\mathcal{L}$. This settles, in a much stronger form, the conjectures of D. Mayer and G. Roepstorff (1988), A. J. MacLeod (1992), Ph. Flajolet and B. Vallée (1995), also posed by several other authors. Further, we find an exact series for the eigenvalues, which also gives the canonical decomposition of trace formulas due to D. Mayer (1976) and K. I. Babenko (1978). This crystallizes the contribution of each individual eigenvalue in the trace formulas.

CONTENTS

1. THE OPERATOR AND THE CONJECTURES

1.1. **Introduction.** Let $\mathbb{D}$ be the disc $\{z \in \mathbb{C} : |z-1| < \frac{3}{2}\}$. Let $\mathbf{V}$ be the Banach space of functions which are analytic in $\mathbb{D}$ and are continuous in its closure, with the supremum norm. *The*

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Perron-Frobenius, or *the transfer operator for the Gauss' continued fraction map*, also called *the Gauss-Kuzmin-Wirsing operator*, is defined for functions $f \in \mathbf{V}$ by [27, 34, 35, 42, 43, 51, 31]

$$\mathcal{L}[f(t)](z) = \sum_{m=1}^{\infty} \frac{1}{(z+m)^2} f\Big(\frac{1}{z+m}\Big). \tag{1}$$

Our chief interest is the point spectrum of this operator.

As was shown in [8, 9], the operator $\mathcal{L}$ is then of trace class and is nuclear of order 0. Thus, it possesses the eigenvalues λ_n, $n \in \mathbb{N}$, which are real numbers, $|\lambda_n| \geq |\lambda_{n+1}|$, $\lambda_1 = 1$, and

$$\sum_{n=1}^{\infty} |\lambda_n|^{\epsilon} < +\infty \text{ for every } \epsilon > 0.$$

In fact, the Hilbert-Schmidt operator $\mathcal{K}$, defined by

$$\mathcal{K}[u(t)](x) = \int_0^{\infty} \frac{J_1(2\sqrt{xy})}{\sqrt{(e^x-1)(e^y-1)}} u(y)\,\mathrm{d}y,$$

for u belonging to the Hilbert space $L^2(\mathbb{R}_+, m)$, $\mathrm{d}m(y) = \frac{y}{e^y-1}\,\mathrm{d}y$, has the same point spectrum, counting algebraic multiplicities [8, 42, 43]; here $J_1(\star)$ is the Bessel function with an index 1. Numerically, eigenvalues are presented in the Table 1. Pascal Sebah [48] provided high-precision numerical calculations. We note that sometimes other authors present not accurate numerical results (concerning 10^{th} or higher significant digits).

n	$(-1)^{n+1}\lambda_n$	n	$(-1)^{n+1}\lambda_n$	n	$(-1)^{n+1}\lambda_n \cdot 10^6$
1	1.0000000000000000	9	0.0002441314655245158	17	0.1010905532214992
2	0.3036630028987326	10	0.00009168908376859330	18	0.03834969795026564
3	0.1008845092931040	11	0.00003451654616385425	19	0.01455613838668023
4	0.03549615902165984	12	0.00001301769787702303	20	0.005527567937997608
5	0.01284379036244026	13	0.000004916782302464491	21	0.002099913582972687
6	0.004717777511571031	14	0.000001859307351509042	22	0.0007980457682720196
7	0.001748675124305511	15	0.0000007038113430870398	23	0.0003033862949098575
8	0.0006520208583205029	16	0.0000002666413434479564	24	0.0001153695418144668

TABLE 1. The eigenvalues λ_n for $1 \leq n \leq 24$, with 16 significant digits.

The transfer operator $\mathcal{L}$ arises from the Gauss map $F(x) = \{1/x\}$, $x \in (0,1]$, $F(0) = 0$, in the following way: for functions $f \in L^1[0,1]$ it is defined by

$$\mathcal{L}[f(t)](z) = \sum_{y \in F^{-1}(z)} \frac{1}{|F'(y)|} \cdot f(y),$$

which agrees with (1). Let further $F^{(1)} = F$, and $F^{(k)} = F \circ F^{(k-1)}$ for $k \geq 2$. As is now well-known, due to important contributions by C. F. Gauss, R. O. Kuzmin, P. Lévy, E. Wirsing, K. I. Babenko,

K. I. Babenko and S. P. Jur'ev, D. Mayer, we have

$$\mu(a\in[0,1]:F^{(k)}(a)<z)=\frac{\log(1+z)}{\log 2}+\sum_{n=2}^{\infty}\lambda_n^k\Phi_n(z). \tag{2}$$

Here $\mu(\star)$ stands for the Lebesgue measure, and for each $n\geq 2$, the eigenfunction $\Phi_n(z)$ is defined in the cut plane $\mathbb{C}\setminus(-\infty,-1]$, it satisfies the conditions $\Phi_n(0)=\Phi_n(1)=0$, the regularity condition

$$\sup_{\Re(z)\geq-\frac{1}{2}}\big|(z+1)U(z)\big|<+\infty. \tag{3}$$

where $U(z)=\Phi'_n(z)$, and the functional equation

$$\Phi_n(z+1)-\Phi_n(z)=\frac{1}{\lambda_n}\cdot\Phi_n\Big(\frac{1}{z+1}\Big).$$

Thus, $\Phi_1(z)=\frac{\log(1+z)}{\log 2}$, $\lambda_1=1$. The eigenfunctions of $\mathcal{L}$ are then given by $\Phi'_n(z)$, $n\in\mathbb{N}$. Moreover, every function $U(z)$ which satisfies, for a certain $\lambda\in\mathbb{R}\setminus\{0\}$, the functional equation

$$U(z)=U(z+1)+\frac{1}{\lambda(z+1)^2}U\Big(\frac{1}{z+1}\Big),\quad z\in\mathbb{C}\setminus(-\infty,-1], \tag{4}$$

and the regularity property (3), is the eigenfunction of $\mathcal{L}$ with the eigenvalue λ. More details can be found in [8, 35, 51].

The nature of the eigenvalues λ_n for $n\geq 2$ is unknown. It is widely believed that these constants are unrelated to other most important constants in mathematics. In particular, it is expected that they are neither algebraic numbers nor periods [36]. We remind that *a period* is a value of an absolutely convergent integral of a rational function with algebraic coefficients, over a domain in $\mathbb{R}^n$ given by polynomial inequalities with algebraic coefficients. All periods form a ring $\mathcal{P}$. However, we will soon see in the item (ii) of Theorem 2 that these eigenvalues are deeply related to an *extended ring of periods* $\mathcal{P}[\frac{1}{\pi}]$ in a direct way.

1.2. **Trace formulas.** Now, more that 480 digits of λ_2 have been calculated [12], but one can get rigorous certificates only for the several first few digits of λ_2 and λ_3 [19, 35, 40, 42, 52]. On the other hand, the trace of the operator $\mathcal{L}$ can be given explicitly. As was shown in [41] (see also [17, 18, 37, 42, 43]), we have

$$\begin{aligned}\mathrm{Tr}(\mathcal{L}) &= \sum_{n=1}^{\infty}\lambda_n=\int_0^{\infty}\frac{J_1(2x)}{e^x-1}\,\mathrm{d}x=\sum_{\ell=1}^{\infty}\frac{1}{\xi_\ell^{-2}+1} \\ &= \frac{1}{2}-\frac{1}{2\sqrt{5}}+\frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\binom{2k}{k}\big(\zeta(2k)-1\big) \\ &= 1-\frac{1}{2\sqrt{2}}-\frac{1}{2\sqrt{5}}+\frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\binom{2k}{k}\Big(\zeta(2k)-1-\frac{1}{2^{2k}}\Big),\end{aligned} \tag{5}$$

$$\mathrm{Tr}(\mathcal{L}^2) = \sum_{n=1}^{\infty}\lambda_n^2=\int_0^{\infty}\int_0^{\infty}\frac{J_1(2\sqrt{xy})^2}{(e^x-1)(e^y-1)}\,\mathrm{d}x\,\mathrm{d}y=\sum_{i,j=1}^{\infty}\frac{1}{(\xi_{i,j}\xi_{j,i})^{-2}-1}. \tag{6}$$

Here

$$\begin{aligned}\xi_{\ell} &= \frac{1}{\ell+}\frac{1}{\ell+}\frac{1}{\ell+}\ldots = \frac{\sqrt{\ell^2+4}-\ell}{2}, \quad \xi_{\ell}^{-2}+1 = \frac{\ell^2+4+\ell\sqrt{\ell^2+4}}{2}, \quad \ell\in\mathbb{N},\\ \xi_{i,j} &= \frac{1}{i+}\frac{1}{j+}\frac{1}{i+}\frac{1}{j+}\ldots, \quad (\xi_{i,j}\cdot\xi_{j,i})^{-2}-1 = \frac{D}{2}+\frac{ij+2}{2}\sqrt{D}, \quad D=(ij+2)^2-4, \quad i,j\in\mathbb{N}.\end{aligned}$$

These are quadratic irrationals whose continued fraction expansion are strictly 1 and 2−periodic. That is, they are all fixed points of F and F^2, respectively. The formula for $\mathrm{Tr}(\mathcal{L})$ which involves zeta values was derived in [17, 19]. In Subsection **??** we will derive similar formula for $\mathrm{Tr}(\mathcal{L}^2)$, and this is very useful computationally.

Proposition 1. *We have an identity:*

$$\begin{aligned}\mathrm{Tr}(\mathcal{L}^2) &= \frac{2}{5+3\sqrt{5}}+\frac{1}{3+2\sqrt{3}}+\frac{4}{21+5\sqrt{21}}+\frac{3}{16+12\sqrt{2}}\\ &+ \sum_{k=2}^{\infty}(-1)^k\binom{2k-2}{k-2}\Big(\zeta^2(k)-1-\frac{2}{2^k}-\frac{2}{3^k}-\frac{3}{4^k}\Big)\\ &= 1.103839653617_{+}.\end{aligned}$$

The kth term of this series is asymptotically equal to $\frac{1}{2}(\frac{4}{5})^k(\pi k)^{-1/2}$.

We have many formulas to calculate zeta values with very high accuracy, and this gives a fast method to calculate $\mathrm{Tr}(\mathcal{L}^2)$. As is clear from [19], the authors of that paper had already derived this formula, though this was not written down explicitly. The importance of this formula is apparent in numerical calculations, when one extracts approximations to the eigenvalues λ_n as eigenvalues of a certain high-order matrix; usually about the order 300-400. While increasing this order some eigenvalues do stabilize, thus it can be guessed that they correspond to some real λ_n, while others ("spurious" ones) tend to disappear. The invalidation of the formula for $\mathrm{Tr}(\mathcal{L}^2)$ also shows that these "spurious" eigenvalues should be ruled out - this fact was noticed by all mathematicians who tried numerically to operate with similar matrices.

In general, for $a_i\in\mathbb{N}$, $1\leq i\leq k$, we put

$$\xi_{a_1,a_2,\ldots,a_k} = [0,\overline{a_1,a_2,\ldots,a_k}\,].$$

This number is a fixed point of $F^{(k)}$. Then we have the fundamental result [43]:

$$\sum_{n=1}^{\infty}\lambda_n^k = \mathrm{Tr}(\mathcal{L}^k) = \sum_{i_1,i_2,\ldots,i_k=1}^{\infty}\Big[\prod_{s=1}^{k}\xi^{-2}_{i_s,i_{s+1},\ldots,i_k,i_1,\ldots,i_{s-1}}-(-1)^k\Big]^{-1}. \tag{7}$$

We will see in Subsection **??** that for the number in the square brackets, denoted there by ϱ, we have a very simple formula. Indeed, let $p=\xi_{i_1,i_2,\ldots,i_k}$, and

$$\frac{ap+b}{cp+d}=p, \text{where } a,b,c,d\in\mathbb{N}_0, \quad \begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} P_{k-1} & P_k\\ Q_{k-1} & Q_k\end{pmatrix},$$

where $\frac{P_s}{Q_s}$ is the sth convergent to p, $\frac{P_1}{Q_1} = \frac{1}{i_1}$. Then

$$\prod_{s=1}^{k} \xi^{-2}_{i_s,i_{s+1},\ldots,i_k,i_1,\ldots,i_{s-1}} - (-1)^k = \frac{D}{2} + \frac{a+d}{2}\sqrt{D}, \quad D = (a+d)^2 - 4(-1)^k.$$

This, of course, does not depend on the cyclic permutation of $\{i_1, i_2, \ldots, i_k\}$, since the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is then replaced by a similar matrix. Indeed,

$$\begin{pmatrix} P_{k-1} & P_k \\ Q_{k-1} & Q_k \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & i_1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & i_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & i_k \end{pmatrix},$$

and the fact becomes obvious.

Without going into details, we note that these trace formulas and the field itself is deeply and intricately related to the Selberg zeta function [32], the Riemann zeta function, Maass wave forms and modular forms for the full modular group [2, 37, 39, 44, 52] - see Subsection **??**.

1.3. **Previous conjectures.** Throughout this paper, we fix the notation

$$\phi = \frac{1+\sqrt{5}}{2}.$$

The constants λ_n have received a considerable amount of attention in recent decades. Nevertheless, there remained three outstanding unresolved problems. No theoretical progress was made towards any of them, only a computational one. As was said before, we henceforth arrange the eigenvalues according to their absolute value:

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots.$$

Of course, in case $\lambda_n = \pm\lambda_{n+1}$ for some n, this arrangement is not uniquely defined. Despite this, we have

Conjecture. *The following three statements are true:*

i) Simplicity. *The eigenvalues are simple, $|\lambda_n|$ strictly decreases.*
ii) Sign. *The eigenvalues have alternating sign: $(-1)^{n+1}\lambda_n > 0$.*
iii) Ratio. *There exists $\lim\limits_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = -\frac{3+\sqrt{5}}{2} = -\phi^2$.*

The first conjecture can be attributed to Babenko [8] and Mayer and Roepstorff [43], the second - to Mayer and Roepstorff too, also reiterated by MacLeod [40]. The last was raised by MacLeod (only with a constant ≈ -2.6), and seconded by Flajolet and Vallée [17, 19, 20, 21], with the constant $-\phi^2$. Several other authors also claimed to believe these conjectures, or state that "for the time being [2014], no way for obtaining [formula for the eigenvalues] can be expected" [30].

The ratio conjecture has the following explanation. The spectrum of the operator

$$\mathcal{L}_0[f(t)](z) = \frac{1}{(z+1)^2} f\Big(\frac{1}{z+1}\Big), \quad f \in \mathbf{V},$$

is given by $\lambda_n^0 = (-1)^{n+1}\phi^{-2n}$, $n \in \mathbb{N}$, with the corresponding eigenfunction being

$$u_0(n, z) = \frac{(z - \phi^{-1})^{n-1}}{(z + \phi)^{n+1}}. \tag{8}$$

(see Section **??**). It is expected that the terms in (1) for $m \geq 2$ act only as small perturbations to $\mathcal{L}_0$; we will show (using algebra and combinatorics, but no functional analysis) that this is indeed the case. Of course, the "Sign" and "Simplicity" conjectures follow from the "Ratio" conjecture for sufficiently large n, provided it is effective and we can verify these conjectures for the first ititial values of n.

2. Main results

2.1. **Formulation.** It is surprising that in fact the real asymptotics of the sequence λ_n, minding the above remark about $\mathcal{L}_0$, is the simplest imaginable from the ratio conjecture.

Theorem 1 (Asymptotics)**.** *We have the formula*

$$\begin{aligned}(-1)^{n+1}\lambda_n &= \phi^{-2n} + C \cdot \frac{\phi^{-2n}}{\sqrt{n}} + d(n) \cdot \frac{\phi^{-2n}}{n}, \\ \text{where the constant } C &= \frac{\sqrt[4]{5} \cdot \zeta(3/2)}{2\sqrt{\pi}} = 1.1019785625880999_+;\end{aligned}$$

here $\zeta(\star)$ is the Riemann zeta function, fand the function $d(n)$ is bounded.

Based on high precision numerical computations of P. Sebah [48], we calculated the function $d(n)$ in Table 2. Since $\frac{\phi^{-300}}{150} \approx 0.13 \cdot 10^{-64}$, this table overwhelmingly convinces in the validity of Theorem 1.

Indeed, suppose we change the constant "2" in the denominator of the fraction which defines C with 1.9 and 2.1, respectively. The values of the so obtained functions $d_{1.9}(n)$ and $d_{2.1}(n)$ for $n = 148, 149$, and 150 would then be, respectively,

$$-0.346715, \quad -0.349104, \quad -0.351485; \text{ and } 0.997259, \quad 0.999403, \quad 1.001539.$$

In the first case, the value for $d_{1.9}(n)$ decreases by approx. 6.8‰ (per mille) at each step, and in the second $d_{2.1}(n)$ increases by approx. 2.1‰. Whereas for the correct constant 2, we see the decrease of $d(n) = d_{2.0}(n)$ by only 0.025‰ at each step.

n	$d(n)$	n	$d(n)$	n	$d(n)$	n	$d(n)$
1	0.51605	5	0.43430	40	0.36268	130	0.359061
2	0.60424	10	0.38504	50	0.36159	148	0.358871
3	0.52221	20	0.36884	70	0.36040	149	0.358862
4	0.46629	30	0.36460	100	0.35952	150	0.358852

TABLE 2. The function $d(n)$.

Theorem 1 is much stronger than the ratio conjecture. Seemingly, the apparent distance from $\lambda_1\phi^2 = 2.61803_+$ to 1 prevented other mathematicians from posing a much stronger conjecture on the asymptotics. Nevertheless, for example, we have: $|\lambda_{150}|\cdot\phi^{300}=1.0923_{+}$.

Our second result gives the much more refined structure of the eigenvalues. Let $P^{(\alpha,\beta)}_{m}(x)$ stand for the classical Jacobi polynomials [49]; see Subsection A.1 for more details.

Theorem 2 (Arithmetic and decomposition of trace formulas)**.** *The following holds.*
(i) There exist functions $W_{v}(\mathbf{X})$, $v\geq 0$, defined by $W_{0}(\mathbf{X})=1$, $W_{1}(\mathbf{X})=\frac{5}{4}\cdot\phi^{-2\mathbf{X}}P^{(0,1)}_{\mathbf{X}-1}(3/2)$, and then by a certain explicit recurrence - this will be given later, see (22) - such that

$$(-1)^{n+1}\lambda_{n}=\phi^{-2n}\sum_{\ell=0}^{\infty}W_{\ell}(n).$$

For the functions $W_{\ell}(n)$ we have an asymptotic formula

$$W_{\ell}(n)=\frac{\sqrt[4]{5}}{2\sqrt{\pi}\cdot\ell^{3/2}\sqrt{n}}+\frac{B}{\ell^{3/2}n}\text{ for } n,\ell\geq 1,$$

where the function $B=B(\ell,n)$ is bounded.

(ii) This decomposition is compatible with (7) and gives the decomposition of trace formulas for the powers of $\mathcal{L}$: for the first, the second, and the third powers we have, respectively,

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n+1}\phi^{-2n}W_{\ell-1}(n) &= \frac{1}{\xi_{\ell}^{-2}+1}, \quad \ell\geq 1,\\ \sum_{i+j=\ell}\sum_{n=1}^{\infty}\phi^{-4n}W_{i-1}(n)W_{j-1}(n) &= \sum_{i+j=\ell}\frac{1}{(\xi_{i,j}\xi_{j,i})^{-2}-1}, \quad \ell\geq 2,\\ \sum_{i+j+k=\ell}\sum_{n=1}^{\infty}(-1)^{n+1}\phi^{-6n}W_{i-1}(n)W_{j-1}(n)W_{k-1}(n) &= \sum_{i+j+k=\ell}\frac{1}{(\xi_{i,j,k}\xi_{j,k,i}\xi_{k,i,j})^{-2}+1}, \quad \ell\geq 3.\end{aligned}\tag{9}$$

Analogously for higher powers of $\mathcal{L}$, for $\mathcal{L}^{k}$.

(iii)$\star$[1] Further, let $\mathbf{a}=\{\ell_{1},\ell_{2},\ldots,\ell_{s}\}$, $\ell_{i}\in\mathbb{N}$. Let us define

$$\Omega_{\mathbf{a}}(w)=\sum_{n=1}^{\infty}\Big(\prod_{i=1}^{s}W_{\ell_{i}}(n)\Big)w^{n},\quad |w|<1.$$

Then all $\Omega_{\mathbf{a}}(w)=\Omega_{\ell_{1},\ell_{2},\ldots,\ell_{s}}(w)$ are arithmetic functions.

(iv) The solution to (4) for the nth eigenvalue, the nth eigenfunction, can be canonically normalized, so that there exists a sequence of structural constants $\{U_{n}(\phi^{-1}),n\in\mathbb{N}\}$.

So, the fundamental trace formulas (7) are split into a countable number of formulas each. Moreover, we will see in the Subsection 3.4 that, due to combinatoric reasons alone, one further refined decomposition occurs. Concerning the item (iv), one canonical normalization of U_{n} comes from (2), but it is ineffective. The normalization we propose is effective and the most natural in deriving

[1] This part will be explained, proved and expanded in [1].

formula in the item (i), as will be soon clear; see Proposition 2.

The simplest examples of (iii) are

$$\begin{aligned} \Omega_{\varnothing}(w) &= \frac{w}{1-w}, \\ \Omega_{1}(w) &= \frac{\phi^2+w}{2\big((\phi^4-w)(1-w)\big)^{1/2}}-\frac{1}{2}. \end{aligned} \tag{10}$$

So, in the trace formulas now we are able to crystallize the contribution of each individual eigenvalue. Thus, this defines an infinite matrix

$$\Big((-1)^{n+1}\phi^{-2n}W_{\ell-1}(n)\Big)_{n,\ell=1}^{\infty},$$

whose elements in rows add up to eigenvalues, elements in columns add up to $(\xi_{\ell}^{-2}+1)^{-1}$, and the sum of all real numbers in the matrix is equal to $\mathrm{Tr}(\mathcal{L})$. At the same time we are able to define a unique two variable function which governs the whole collection of eigenvalues and trace formulas:

$$\mathfrak{G}(w,\,\omega)=\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}w^n\,\omega^{\ell-1}W_{\ell-1}(n)=\sum_{\ell=0}^{\infty}\Omega_{\ell}(w)\,\omega^{\ell},\quad |\,\omega|\leq 1,\quad |w|<1.$$

For example,

$$-\mathfrak{G}(-\phi^{-2},\,\omega)=\sum_{n=1}^{\infty}\lambda_n(\,\omega)=\mathrm{Tr}(\mathcal{L}_{\omega})=\sum_{\ell=1}^{\infty}\frac{\omega^{\ell-1}}{\xi_{\ell}^{-2}+1}=\int_0^{\infty}\frac{J_1(2x)}{e^x-\,\omega}\,\mathrm{d}x.$$

The operator $\mathcal{L}_{\omega}$ will be soon defined in Subsection 2.2. It is important to observe that Theorem 2, for the first time, gives rigorous certificates (though considerable computational time and space resources are needed) to calculate numerically the first few digits of an eigenvalue with any index.

2.2. **The approach.** Our method of proving these results is constructive. Consider a function $f(\,\omega,z)$, which is analytic in ω for $|\,\omega|\leq 1$, and for every such ω, $f(\,\omega,z)\in\mathbf{V}$ as a function in z. Denote the set of all such functions by $\widetilde{\mathbf{V}}$. Then

$$\|f\|_{\widetilde{\mathbf{V}}}=\sup_{\substack{|\omega|\leq 1\\ |z-1|\leq\frac{3}{2}}}\big|f(\omega,z)\big|$$

makes $\widetilde{\mathbf{V}}$ into a Banach space. Let us define the operator $\mathcal{L}_{\,\omega}:\widetilde{\mathbf{V}}\mapsto\widetilde{\mathbf{V}}$ by

$$\mathcal{L}_{\,\omega}[f(\,\omega,t)](z)=\sum_{m=1}^{\infty}\frac{\omega^{m-1}}{(z+m)^2}f\Big(\,\omega,\frac{1}{z+m}\Big). \tag{11}$$

If $G(\,\omega,z)$ is an eigenfunction of this operator, then a function $G(\,\omega,z)$ is defined up to a "scalar" multiple, which (in this case) is an arbitrary function in ω, holomorphic for $|\,\omega|\leq 1$. Thus, if $\lambda(\,\omega)$ is an eigenvalue of this operator, then

$$\lambda(\,\omega)G(\,\omega,z)=\,\omega\lambda(\,\omega)G(\,\omega,z+1)+\frac{1}{(z+1)^2}\cdot G\Big(\,\omega,\frac{1}{z+1}\Big),\quad z\in\mathbb{C}\setminus(-\infty,-1],\quad |\,\omega|\leq 1. \tag{12}$$

We note that though $G(\omega,z)$ is initially defined for $|z-1|<\frac{3}{2}$, eigenfunctions automatically extend to the region $\{\mathbb{C}\setminus(-\infty,-1]\}$.

For a two variable analytic function $g(\omega,x)$, let $g(\omega,x)[x^m]=\beta_m(\omega)$ is the coefficient at x^m in a Taylor series expansion of $g(\omega,x)$ around $x=0$. All our results follow from the following explicit construction.

Proposition 2. *For every $n\in\mathbb{N}$, there exists the unique analytic function $\lambda_n(\omega)$ and the unique analytic function $G_n(\omega,z)$, $|\omega|\leq 1$, $\Re z>-\frac{1}{2}$, whose Taylor coefficients in the variable ω are explicitly constructable (see Section* **??***), which satisfies in conjunction the functional equation (12) and the regularity condition (3) (uniformly in ω), with the following initial value and normalization properties:*

i)

$$\lambda_n(0)=(-1)^{n+1}\phi^{-2n},$$

ii)

$$G_n(0,z)=\frac{(z-\phi^{-1})^{n-1}}{(z+\phi)^{n+1}},$$

iii)

$$\frac{5}{(1-x)^2}G_n\Big(\omega,\frac{x\phi+\phi^{-1}}{1-x}\Big)[x^{n-1}]=1;$$

here $1=1\,\omega^0+0\,\omega^1+0\,\omega^2+\cdots$ is a constant function in ω.

In terms of Section **??**, the last item will correspond to the choice $q_0^{(n)}=1$, $q_v^{(n)}=0$ for $v\geq 1$. We include the second property for clarity only, since it is the consequence of the first property, the regularity condition, and the three term functional equation (12). Thus, for every n, we construct explicitly $\lambda_n=\lambda_n(1)$ and the solution $U_n(z)=G_n(1,z)$, therefore canonically normalized which together satisfy (4). The existence of this canonical normalization, alternative to the one implied by (2), is also a novel feature in the theory of the transfer operator $\mathcal{L}$. This means that for each eigenvalue λ_n, there exists the canonical solution to (4); for example, this reveals a set of new structural constants $\{U_n(\phi^{-1}):n\in\mathbb{N}\}$, as claimed by the item (iv) of Theorem 2. The specific choice of $z=\phi^{-1}$ will become clear from Subsection (**??**).

Comparing trace formulas we will see that there are no other eigenvalues and eigenfunctions apart from those explicitly constructed. *A priori*, eigenvalue $\delta(\omega)$ of some operator $T_\omega:\widetilde{\mathbf{V}}\mapsto\widetilde{\mathbf{V}}$, which is analytic in ω, can be expressed as a *Puiseux series*, thus containing fractional powers of ω:

$$\delta(\omega)=\sum_{k=0}^{\infty}a_k\,\omega^{k/L},\quad a_k\in\mathbb{C},\ \text{for some } L\in\mathbb{N}.$$

Corollary. *Puiseux series for the eigenvalues and eigenfunctions of (12) contain only integral powers of ω.*

Thus, collecting everything together, we obtain

Corollary. *All three claims of the Conjecture are true.*

Note that we prove the *eigenvalue simplicity conjecture* independently of the asymptotic result: the proof of simplicity, as well as the decomposition formulas are based on Proposition 2, but follow independently. Indeed, this proposition shows that for one particular eigenvalue only one eigenfunction can be constructed, and there are no other, since that would invalidate the trace formulas. For the sign conjecture there is no direct proof, so we must rely on computer resources and explicit bounds for the function $B(\ell, n)$ from the Theorem 2.

3. Examples and decompositions

3.1. **Initial cases of decomposition.** In the next examples k always means which power of the operator $\mathcal{L}$ we are looking at, and ℓ - the sum of partial quotients in the period of the quadratic irrational.

The decomposition identities of Theorem 1, (ii) can be checked to hold true for, say, $\ell = 1, 2$ (in the first case which describes $\mathcal{L}^1$), $\ell = 3$ (in the second case which describes $\mathcal{L}^2$), and $\ell = 4$ (in the third case which describes $\mathcal{L}^3$), if we use the generating function for Jacobi polynomials (31):

$$\begin{aligned}
\sum_{n=1}^{\infty}(-1)^{n+1}\phi^{-2n} = \frac{1}{\phi^2+1} &= \frac{1}{\xi_1^{-2}+1} = (-1)^1\Omega_{\varnothing}(-\phi^{-2}),\\
\frac{5}{4}\sum_{n=1}^{\infty}(-1)^{n+1}\phi^{-4n}P_{n-1}^{(0,1)}(3/2) &= \frac{1}{4+2\sqrt{2}} = \frac{1}{\xi_2^{-2}+1} = (-1)^1\Omega_1(-\phi^{-2}),\\
\frac{5}{4}\sum_{n=1}^{\infty}\phi^{-6n}P_{n-1}^{(0,1)}(3/2) &= \frac{1}{6+4\sqrt{3}} = \frac{1}{(\xi_{1,2}\xi_{2,1})^{-2}-1} = (-1)^2\Omega_1(\phi^{-4}),\\
\frac{5}{4}\sum_{n=1}^{\infty}(-1)^{n+1}\phi^{-8n}P_{n-1}^{(0,1)}(3/2) &= \frac{1}{20+6\sqrt{10}} = \frac{1}{(\xi_{1,1,2}\xi_{1,2,1}\xi_{2,1,1})^{-2}+1} = (-1)^3\Omega_1(-\phi^{-6}).
\end{aligned}$$

MAPLE re-confirms this, too. In general, we will see in Subsection **??**, and this can be calculated directly from (10), that

$$(-1)^k\Omega\Big((-1)^k\phi^{-2k}\Big) = \frac{1}{\psi_k}, \quad \psi_k = 2F_{k+1}^2 - 2(-1)^k + 2F_{k+1}\sqrt{F_{k+1}^2-(-1)^k}. \tag{13}$$

Here F_k is the standard Fibonacci sequence with the (standard) seed values $F_1 = F_2 = 1$.

$\boxed{\ell = 3,\ k = 1.}$ Next, though we do not have explicit values for $W_2(n)$ in a closed form (but see (27) and Section 7), but numerical values can be calculated easily to a very high precision. So, in particular, the first identity of (ii) for $\ell = 3$, minding the values in Table 4 and further calculations, give the correct anticipated value, and this shows that our method indeed works! Thus, MAPLE does indeed confirm that

$$\sum_{n=1}^{\infty}(-1)^{n+1}W_2(n)\phi^{-2n} = 0.0839748528310781_+ = \frac{2}{13+3\sqrt{13}} = \frac{1}{\xi_3^{-2}+1}.$$

$\boxed{\ell = 4,\, k = 2.}$ A still more interesting example occurs in the second identity of (ii) in case $\ell = 4$. Numerically, we have:

$$\begin{aligned} A &= \sum_{n=1}^{\infty} W_2(n)\phi^{-4n} = 0.0428848639793538_{+}, \\ B &= \sum_{n=1}^{\infty} W_1^2(n)\phi^{-4n} = 0.0356498091111648_{+}, \end{aligned}$$

$$\begin{aligned} C &= \frac{2}{21+5\sqrt{21}} = \frac{1}{(\xi_{1,3}\xi_{3,1})^{-2}-1} = 0.0455447255899809_{+}, \\ D &= \frac{1}{16+12\sqrt{2}} = \frac{1}{(\xi_{2,2}\xi_{2,2})^{-2}-1} = 0.0303300858899106_{+}. \end{aligned}$$

And thus, we get the correct identity (compare (19) and (20) below for $k = 2$)

$$2A + B = 2C + D = 0.1214195370698725_{+}. \tag{14}$$

As can be anticipated, this identity apparently does not decompose into two identities, since there is no *a priori* reason why A or B should be algebraic numbers. In fact, we know the formula (32), where B is expressed in terms of complete elliptic integrals of the first and the third kind. In our case $w = \phi^{-4}$. So, $z = \frac{\phi^2}{3}$, $z\sqrt{w} = \frac{1}{3}$, $zw = \frac{\phi^{-2}}{3}$, and consequently,

$$4\pi \cdot \Big(B + \frac{1}{2}\Big) = \frac{4\sqrt{5}}{3} \cdot K\Big(\frac{1}{3}\Big) + \frac{2}{3} \cdot \Pi\Big(\frac{\phi^2}{3}, \frac{1}{3}\Big) - \frac{2}{3} \cdot \Pi\Big(\frac{\phi^{-2}}{3}, \frac{1}{3}\Big).$$

We recall that

$$\Pi(\nu, k) = \int_0^1 \frac{\mathrm{d}t}{(1-\nu t^2)\sqrt{(1-t^2)(1-k^2t^2)}}, \quad K(k) = \Pi(0, k).$$

So, according to the notion of Zagier-Kontsevitch [36], which we mentioned in the end of Subsection 1.1, the number B is *arithmetic* in the sense that it belongs to the extended ring of periods $\widehat{\mathcal{P}} = \mathcal{P}[\frac{1}{\pi}]$, and then so does A, since algebraic numbers are also periods. In general, in the next subsection we will see that

$$\Omega_2\Big((-1)^k\phi^{-2k}\Big) \in \widehat{\mathcal{P}} \text{ for } k \in \mathbb{N}.$$

This is exactly what we mean by the item (iii) in Theorem 2. An alternative approach is presented in Section 7.

$\boxed{\ell=5,\ k=3.}$ Further, the last identity of (ii) in case $\ell=5$ gives the following. Let us define

$$\begin{aligned} E &= \sum_{n=1}^{\infty}(-1)^{n+1}W_2(n)\phi^{-6n} = 0.0144368544431652_+, \\ F &= \sum_{n=1}^{\infty}(-1)^{n+1}W_1^2(n)\phi^{-6n} = 0.0123983653921924_+, \\ G &= \frac{1}{34+8\sqrt{17}} = \frac{1}{(\xi_{1,1,3}\xi_{1,3,1}\xi_{3,1,1})^{-2}+1} = 0.0149287499273340_+, \\ H &= \frac{2}{85+9\sqrt{85}} = \frac{1}{(\xi_{1,2,2}\xi_{2,2,1}\xi_{2,1,2})^{-2}+1} = 0.0119064699080236_+. \end{aligned}$$

Then

$$E+F=G+H=0.0268352198353576_+.$$

In fact, the numbers G and H are special cases of the numbers which will be investigated in Subsections **??** and **??**, respectively: $1/G$ is given by the formula (**??**) ($N=3$, $k=3$), and $1/H$ - by the formulas (**??**) and (**??**) ($k=3$, $r=2$). As noted above, $E,F\in\widehat{\mathcal{P}}$.

$\boxed{\ell=6,\ k=4.}$ This is a final example from the ones with $\ell-k=2$. We will see now the sum of three quadratic irrationals. Indeed, let

$$\begin{aligned} I &= \sum_{n=1}^{\infty}W_2(n)\phi^{-8n} = 0.0057706896109551_+, \\ J &= \sum_{n=1}^{\infty}W_1^2(n)\phi^{-8n} = 0.0048993082411535_+, \end{aligned}$$

$$\begin{aligned} K &= \frac{2}{165+13\sqrt{165}} = \frac{1}{(\xi_{1,1,1,3}\xi_{1,1,3,1}\xi_{1,3,1,1}\xi_{3,1,1,1})^{-2}-1} = 0.0060243137049899_+, \\ L &= \frac{2}{221+15\sqrt{221}} = \frac{1}{(\xi_{2,2,1,1}\xi_{2,1,1,2}\xi_{1,1,2,2}\xi_{1,2,2,1})^{-2}-1} = 0.0045045959723434_+, \\ M &= \frac{1}{96+56\sqrt{3}} = \frac{1}{(\xi_{2,1,2,1}\xi_{1,2,1,2}\xi_{2,1,2,1}\xi_{1,2,1,2})^{-2}-1} = 0.0051814855409225_+. \end{aligned}$$

Again, we used formulas (**??**), (**??**) and (**??**). MAPLE confirms that

$$2I+3J=2K+2L+M=0.0262393039453711_+.$$

$\boxed{\ell=4,\ k=1,\text{ and }\ell=5,\ k=2.}$ Since there are infinitely many identities involving the three functions W_1, W_2, and W_3, we will confine to the last three examples, now involving W_3. First, we will check the validity of the first two identities of (ii) for $\ell=4$ and $\ell=5$, respectively. Using the values in Table 5 and further calculations, we thus get the correct value:

$$\sum_{n=1}^{\infty}(-1)^{n+1}W_3(n)\phi^{-2n} = 0.052786404500042_+ = \frac{1}{10+4\sqrt{5}} = \frac{1}{\xi_4^{-2}+1}.$$

Next, let us define

$$
\begin{aligned}
N &= \sum_{n=1}^{\infty} W_3(n)\phi^{-4n} = 0.0268901808819348_{+},\\
O &= \sum_{n=1}^{\infty} W_1(n)W_2(n)\phi^{-4n} = 0.0198376845022980_{+},
\end{aligned}
$$

$$
\begin{aligned}
P &= \frac{1}{(\xi_{1,4}\xi_{4,1})^{-2}-1} = \frac{1}{16+12\sqrt{2}} = 0.0303300858899106_{+},\\
Q &= \frac{1}{(\xi_{2,3}\xi_{3,2})^{-2}-1} = \frac{1}{30+8\sqrt{15}} = 0.0163977794943222_{+}.
\end{aligned}
$$

Note that, according to the formula in Subsection **??**, $\xi_{2,2}\xi_{2,2}=\xi_{1,4}\xi_{4,1}$; and so $D=P$. Then we get the correct identity:

$$
N+O=P+Q=0.0467278653842328_{+}.
$$

$\boxed{\ell=6,\ k=3.}$ This is the final example. Let

$$
\begin{aligned}
R &= \sum_{n=1}^{\infty}(-1)^{n+1}W_3(n)\phi^{-6n} = 0.0090670451347138_{+},\\
S &= \sum_{n=1}^{\infty}(-1)^{n+1}W_1(n)W_2(n)\phi^{-6n} = 0.0069673221927849_{+},\\
T &= \sum_{n=1}^{\infty}(-1)^{n+1}W_1^3(n)\phi^{-6n} = 0.0059673995757538_{+},
\end{aligned}
$$

$$
\begin{aligned}
U &= \frac{1}{52+10\sqrt{26}} = \frac{1}{(\xi_{1,1,4}\xi_{1,4,1}\xi_{4,1,1})^{-2}+1} = 0.0097096621545399_{+},\\
V &= \frac{1}{74+12\sqrt{37}} = \frac{1}{(\xi_{1,2,3}\xi_{2,3,1}\xi_{3,1,2})^{-2}+1} = 0.0068030380839281_{+},\\
W &= \frac{1}{74+12\sqrt{37}} = \frac{1}{(\xi_{1,3,2}\xi_{3,2,1}\xi_{2,1,3})^{-2}+1} = 0.0068030380839281_{+},\\
X &= \frac{1}{100+70\sqrt{2}} = \frac{1}{(\xi_{2,2,2}\xi_{2,2,2}\xi_{2,2,2})^{-2}+1} = 0.0050252531694167_{+}.
\end{aligned}
$$

Quadratic irrationals can be calculated directly, but it is most convenient to use formula (**??**), $s=1$. The correct identity, confirmed numerically, is

$$
3R+6S+T=3U+3V+3W+X=0.0074972468136605_{+}.
$$

3.2. **Overlapping values.** One interesting question which arises while inspecting the above examples and Theorem 2 is as follows.

Some values of $\Omega_{\mathbf{a}}(w)$ at $w=(-1)^k\phi^{-2k}$ are left out. Namely, only values for $k\geq s$ occur in Theorem 2, where $\mathbf{a}=\{\ell_1,\ldots,\ell_s\}$. Hence, we pose the following problem, which we intend to investigate in the second part [1].

Problem 1. *Describe the arithmetic meaning of the values*

$$\Omega_{\ell_{\ell_1,\ell_2,\ldots,\ell_s}}\Big((-1)^k\phi^{-2k}\Big),\quad 1\leq k<s.$$

For example, let

$$\begin{aligned} Y &= \sum_{n=1}^{\infty}(-1)^{n+1}W_1^3(n)\phi^{-2n}=0.0375599269123177_{+},\\ Z &= \sum_{n=1}^{\infty}(-1)^{n+1}W_1(n)W_2(n)\phi^{-2n}=0.0426901777719431_{+}. \end{aligned}$$

However, MAPLE package `IntegerRelations` and the PSLQ algorithm does not seem to find that the number $aY+bZ$ for small $a,b\in\mathbb{Z}$ might be an algebraic number of degree 4; this includes the sum of two quadratic irrationals. It might happen that some combination of R and S is the sum of three quadratic irrationals, like in the example $\{\ell=6,\,k=4\}$ above. However, even for the sum of two quadratic irrationals, the coefficients of the 4th degree polynomial are in the range of 10.000, hence much more powerful numerical calculations are needed to detect any algebraicity.

3.3. **Arithmetic of decomposition formulas.** The main Theorem 2 is derived while investigating the $g-$coefficients of a function $U(z)$ which satisfies the functional equation (4) and the regularity property (3); see Subsection **??** and Sections **??** and 4. In fact, there exists the second way to get information on the functions $W_\ell(n)$, and it is based on the trace formulas of D. Mayer. This second method of approach towards the whole problem of determining arithmetic and structure of the eigenvalues reduces to investigation of high powers of the operator $\mathcal{L}_\omega$ (see (11) and Section 4) and then at inspecting the small powers of ω.

For example, take the k^{th} power of $\mathcal{L}_\omega$. First, we have [41, 43]:

$$\mathrm{Tr}(\mathcal{L}_\omega^k)=\sum_{i_1,i_2,\ldots,i_k=1}^{\infty}\omega^{i_1+i_2+\cdots+i_k-k}\Big[\prod_{s=1}^{k}\xi^{-2}_{i_s,i_{s+1},\ldots,i_k,i_1,\ldots,i_{s-1}}-(-1)^k\Big]^{-1}. \tag{15}$$

On the other hand, by our result (**??**),

$$\begin{aligned} \mathrm{Tr}(\mathcal{L}_\omega^k) &= \sum_{n=1}^{\infty}\lambda_n^k(\omega)=\sum_{n=1}^{\infty}(-1)^{nk+k}\phi^{-2nk}\Big(\sum_{\ell=0}^{\infty}\omega^\ell\cdot W_\ell(n)\Big)^k\\ &= \sum_{\ell=0}^{\infty}\omega^\ell\sum_{n=1}^{\infty}(-1)^{nk+k}\phi^{-2nk}\sum_{j_1+\cdots+j_k=\ell}W_{j_1}(n)\cdots W_{j_k}(n). \end{aligned} \tag{16}$$

Now we proceed with comparing the corresponding coefficients at ω^ℓ. First, the coefficients at ω^0 of (15) is equal to $(\phi^{2k}-(-1)^k)^{-1}$, while the coefficient at ω^0 of (16) is equal to

$$\sum_{n=1}^{\infty}(-1)^{nk+k}\phi^{-2nk}=\frac{1}{\phi^{2k}-(-1)^k},$$

so both values do match.

Now look at the first non-trivial case, namely, the power ω^1. This way we find the function which interpolates a class of quadratic irrationals as follows. Let

$$\xi_{\underbrace{2,1,\ldots,1}_{k}}\cdot\xi_{\underbrace{1,2,\ldots,1}_{k}}\cdots\xi_{\underbrace{1,1,\ldots,2}_{k}}=\Gamma_{2,k}.$$

(For the notation "$\Gamma_{2,k}$", see Subsection **??**). Then (see (10))

$$(-1)^k\Omega_1\Big((-1)^k\phi^{-2k}\Big)=\frac{1}{\Gamma_{2,k}^{-2}-(-1)^k}. \tag{17}$$

In the formula (13), we denoted $\psi_k=\Gamma_{2,k}^{-2}-(-1)^k$. The identity (17) corresponds to the case when the sum of k summands is equal to $k+1$.

3.4. **Refined decomposition.** Now, look at ω^2 of the expansions (15) and (16). This corresponds to the combinatoric case when the sum of k summands is equal to $k+2$. So, this can happen either in the case $3+1+\cdots+1$ (k unities), or one of the cases $2+2+1+\cdots+1$, $2+1+2+1+\cdots+1$ ($k-1$ unities), and so on. Thus, the decomposition of trace formulas gives in this case the following identity:

$$\begin{aligned}&\sum_{n=1}^{\infty}(-1)^{nk+k}\phi^{-2nk}\sum_{j_1+\cdots+j_k=2}W_{j_1}(n)\cdots W_{j_k}(n)\\ =&\sum_{i_1+i_2+\cdots+i_k=k+2}\Big[\prod_{s=1}^{k}\xi^{-2}_{i_s,i_{s+1},\ldots,i_k,i_1,\ldots,i_{s-1}}-(-1)^k\Big]^{-1},\text{ for any }k\in\mathbb{N}.\end{aligned} \tag{18}$$

So, the left hand side, minding the identity $W_0(n)=1$, is equal to

$$\begin{aligned}&(-1)^k k\sum_{n=1}^{\infty}(-1)^{nk}\phi^{-2nk}W_2(n)+(-1)^k\binom{k}{2}\sum_{n=1}^{\infty}(-1)^{nk}\phi^{-2nk}W_1^2(n)\\ =&(-1)^k k\Omega_2\Big((-1)^k\phi^{-2k}\Big)+(-1)^k\binom{k}{2}\Omega_{1,1}\Big((-1)^k\phi^{-2k}\Big).\end{aligned} \tag{19}$$

On the other hand, using the notation of Subsections **??** and **??**, we get that the r.h.s. of (18) is equal to

$$\frac{(-1)^k k}{\Theta_3\Big((-1)^k\phi^{-2k}\Big)}+\frac{1}{2}\sum_{r=2}^{k}\frac{(-1)^k k}{\Pi_r\Big((-1)^k\phi^{-2k}\Big)}, \tag{20}$$

where the first summand corresponds to the case $3+1+\cdots+1$ and all its cyclic permutations, and the second sum - to the cases $2+2+1+\cdots+1$ (Π_2), $2+1+2+1+\cdots+1$ (Π_3), and up to $2+1+\cdots+1+2$ (Π_k). The factor "$\frac{1}{2}$" arises from the fact that any of the two "2"s can be cyclically permute to start the sum. Obviously, $\Pi_r(w)=\Pi_{k-r+2}(w)$. Since the Taylor coefficients of $\Omega_{1,1}(w)$ are equal to $W_1^2(n)=O(n^{-1})$, we can also calculate the asymptotics of the Taylor coefficients (in $w=(-1)^k\phi^{-2k}$) of the function (20), and this gives the exact asymptotics of the function $W_2(n)$.

Moreover, the formula (19) further decomposes as follows.

We see from (19) that the identity which equates it to (20) can be written in the form

$$kA\Big((-1)^k\phi^{-2k}\Big)+k^2B\Big((-1)^k\phi^{-2k}\Big)=kC\Big((-1)^k\phi^{-2k}\Big)+k^2D\Big((-1)^k\phi^{-2k}\Big),\quad k\in\mathbb{N},$$

where $A(w),B(w),C(w),D(w)$ are analytic functions for $|w|<1$. This implies

$$k(A-C)\Big((-1)^k\phi^{-2k}\Big)=k^2(D-B)\Big((-1)^k\phi^{-2k}\Big).$$

Now the assumption that A is not identical to C, after investigating the first non-zero Taylor coefficient of $A(w)-C(w)$, leads to a contradiction. Thus, $A\equiv C$ and $B\equiv D$.

This is the general outline how our second way to calculate the functions $W_\ell(n)$ works, and the first few steps are presented in Subsections **??** and **??**. This method is the main topic of the forthcoming paper [1]. As mentioned, the first way by which we prove all our main results is presented in Section **??**.

4. GENERALIZED OPERATOR

Now, we need to investigate the following question: how we can be sure that $(-1)^{n+1}\Lambda(n)$ gives *all* eigenvalues of $\mathcal{L}$, that there are no "sporadic" ones? The answer is provided by the trace formulas, and for this purpose we will employ a generalized operator $\mathcal{L}_\omega:\widetilde{\mathbf{V}}\mapsto\widetilde{\mathbf{V}},\ \omega\in\mathbb{C},\ |\omega|\le 1$, which has been already defined by (11). Then $\mathcal{L}_1[f(t)](z)=\mathcal{L}[f(t)](z)$. If $G(\omega,z)$ is the eigenfunction of this operator with the eigenvalue $\lambda(\omega)$, then it satisfies the regularity condition (3), and the functional equation (12). In fact, the whole essence of our first method (as said before) can be expressed in the following way.

Let us rewrite the functional equation (12) in terms of a function g, where

$$G(\omega,z)=\frac{1}{(z+\phi)^2}\,g\Big(\omega,\frac{z-\phi^{-1}}{z+\phi}\Big),\quad z=\frac{x\phi+\phi^{-1}}{1-x}.$$

We then obtain

$$\boxed{g(\omega,x)=\frac{\phi^{-2}}{\Lambda(\omega)}g(\omega,-\phi^{-2}x)+\frac{5\,\omega}{(2\phi-x)^2}\,g\Big(\omega,\frac{2x\phi^{-1}+1}{-x+2\phi}\Big).}\tag{21}$$

A function $g(\omega,x)$ is analytic for $|\omega|\le 1$, $|x|\le 1$ (excluding only $x=1$ which corresponds to $z=\infty$), $\sup_{|x|\le 1}|(1-x)g(\omega,x)|<+\infty$. This is just a reformulation of a regularity property for $G(\omega,z)$. Note that since $G(\omega,z)$ is analytic in the cut $z-$plane $\mathbb{C}\setminus(-\infty,-1]$, then $g(\omega,x)$ is analytic in the cut $x-$plane $\mathbb{C}\setminus(-\infty,-\phi^2]\cup[1,\infty)$.

We will explicitly construct n solutions to the above functional equation, given by

$$g_n(\omega,x)=\sum_{\ell=0}^{\infty}\sum_{i=1}^{\infty}q_\ell^{(i)}(n)\,\omega^\ell x^{i-1},\quad \Lambda_n(\omega)=\frac{(-1)^{n+1}\phi^{-2n}}{1-\sum\limits_{\ell=1}^{\infty}Y_\ell(n)\,\omega^\ell}.$$

comparing coefficients at $\omega^v x^{j-1}$ for $v\geq 1$, $j\geq 1$o, f the above functional equation we obtain

$$\boxed{q_v^{(j)}(n)\frac{\lambda_n-\lambda_j}{\lambda_j}+\sum_{r=1}^{v}Y_r(n)\,q_{v-r}^{(j)}(n)=\frac{\lambda_n}{\lambda_j}\sum_{i=1}^{\infty}q_{v-1}^{(i)}(n)\,K(j,i),\quad v\geq 0,\quad j\geq 1.} \tag{22}$$

4.1. **The recurrence.** Hoiw the recurrence (22) works? Let $v=1$ in (22). Then this reads as

$$q_1^{(j)}+q_0^{(j)}Y_1=q_1^{(j)}(-1)^{\mathbf{X}+j}\phi^{2\mathbf{X}-2j}+K(j,\mathbf{X}). \tag{23}$$

When $j=n$, we obtain

$$Y_1=K(n,n).$$

Note that this will correspond t the second largest contributor to the exact value of λ_n; it is equal to $\phi^{-2n}K(n,n)$, which is of size $\frac{\phi^{-2n}}{\sqrt{n}}$ - this guarantees the success of our approach! When $j\neq\mathbf{X}$, (23) gives

$$q_1^{(j)}=\frac{\lambda_n}{\lambda_n-\lambda_j}K(j,n).$$

As we now see, the value of $q_1^{(\mathbf{X})}$ cannot be extracted from (23), and it can be defined arbitrarily. Indeed, choosing another value for $q_\ell^{(\mathbf{X})}$, $\ell\geq 1$, leads to a function $\widetilde{U}(\mathbf{X},z)$ which is different from $U(\mathbf{X},z)$ by a constant factor $\sum_{\ell\geq 0}q_\ell^{(\mathbf{X})}$, provided the last series is absolutely convergent. We therefore always choose $q_\ell^{(\mathbf{X})}=0$ for $\ell\geq 1$. In terms of the ariable ω this means the following. The equation (12) remains valid even if we multiply it by any function $t(\omega)$, analytic for $|\omega|\leq 1$. *A posteriori*, the choice $q_j^{(\mathbf{X})}=0$ gives the canonical normalization of the function U which satisfies (4) for the nth eigenvalue, and this proves item (iv) of Theorem 2.

The initial conditions are

$$q_0^{(j)}(n)=\delta_{j,n},\qquad q_v^{(n)}(n)=0\quad(v\geq 1). \tag{24}$$

In the resonant case $j=n$, this gives

$$Y_v(n)=\sum_{i=1}^{\infty}q_{v-1}^{(i)}(n)\,K(n,i),\qquad v\geq 1. \tag{25}$$

For $j\neq n$, the recurrence (**??**) gives

$$q_v^{(j)}(n)=\frac{\lambda_n}{\lambda_n-\lambda_j}\sum_{i=1}^{\infty}q_{v-1}^{(i)}(n)\,K(j,i)-\frac{\lambda_j}{\lambda_n-\lambda_j}\sum_{r=1}^{v}Y_r(n)\,q_{v-r}^{(j)}(n). \tag{26}$$

We can step-by-step solve these recurrences. In particular,

$$Y_1(n)=K(n,n),\quad Y_2(n)=\sum_{i\neq n}K(n,i)K(i,n)\cdot\frac{\lambda_n}{\lambda_n-\lambda_i}.$$

Proposition 3. *For every fixed $v\geq 1$, as $n\to\infty$, we conjecture that*

$$Y_v(n)\sim\sum_{i_1,\ldots,i_{v-1}>n}K(n,i_1)\,K(i_1,i_2)\cdots K(i_{v-1},n).$$

Moreover, for $j > n$,

$$q_v^{(j)}(n) \sim \frac{\lambda_n}{\lambda_n - \lambda_j} \sum_{i_1,\ldots,i_{v-1}>n} K(j,i_1)\, K(i_1,i_2) \cdots K(i_{v-1},n),$$

while for $j < n$,

$$q_v^{(j)}(n) = B\phi^{-2(n-j)} \sum_{i_1,\ldots,i_{v-1}>n} K(j,i_1)\, K(i_1,i_2) \cdots K(i_{v-1},n).$$

n	$W_1(n)$	$W_1(n)\sqrt{n}$	n	$W_1(n)$	$W_1(n)\sqrt{n}$
1	0.4774575140626314	0.47745751406263	14	0.1132518994300057	0.42374980606847
2	0.3191519488288150	0.45134901449152	15	0.1093746232463436	0.42360609433078
3	0.2525179078163117	0.43737384615885	16	0.1058705808693317	0.42348232347732
4	0.2157725901346884	0.43154518026938	17	0.1026834287993105	0.42337462294015
5	0.1917523856954763	0.42877136926285	18	0.0997680672051097	0.42328006119965
6	0.1744105888874710	0.42721694851264	19	0.0970879075478749	0.42319637764101
7	0.1610997341266682	0.42622983277780	20	0.0946129110363578	0.42312180125288
8	0.1504537306202003	0.42554741270543	21	0.0923181531988805	0.42305492505240
9	0.1416824491533456	0.42504734746005	22	0.0901827554098868	0.42299461723883
10	0.1342909272767424	0.42466519929053	23	0.0881890773429909	0.42293995713338
11	0.1279504805135870	0.42436373560926	24	0.0863220981936739	0.42289018820186
12	0.1224328715287222	0.42411990800860	100	0.0422070239152352	0.42207023915235
13	0.1175738863751953	0.42391867598136	1000	0.0133401861665838	0.42185372697076

TABLE 3. The function $W_1(n)$ for $1 \le n \le 24$, $n = 100, 1000$.

Proposition 8 tells that

$$W_1(n) \sim \frac{5^{1/4}}{2\sqrt{\pi}\sqrt{n}} = \frac{1}{\sqrt{n}} \cdot 0.4218301030679226_{+}.$$

The convergence rate is $n^{-3/2}$. The numerical data is presented in Table 3.

4.2. $v = 2$ **and asymptotic results for** $W_2(n)$**.** As a next step, let us read the recurrence (22) in case $v = 2$. This gives:

$$q_2^{(j)} + q_1^{(j)} W_1 + q_0^{(j)} W_2 = q_2^{(j)} (-1)^{\mathbf{X}+j} \phi^{2\mathbf{X}-2j} + K(j, \mathbf{X}) W_1 + \sum_{i=1}^{\infty} q_1^{(i)} K(j,i).$$

When $j = \mathbf{X}$, this yields

$$W_2 = K^2(\mathbf{X}, \mathbf{X}) + \sum_{i \neq \mathbf{X}} \frac{K(\mathbf{X}, i) K(i, \mathbf{X})}{1 - (-1)^{i+\mathbf{X}} \phi^{2\mathbf{X}-2i}}. \tag{27}$$

For example, this gives

$$\begin{aligned}
W_2(1) &= \frac{25}{16\phi^4} + \sum_{s\neq 1} \frac{25s}{4^{s+1}\phi^4\big(\phi^{2s-2} - (-1)^{s+1}\big)} \\
&= 0.2676322966331297687096221376655991006069231847064740419073 80_+, \\
W_2(2) &= \frac{35^2}{2^8\phi^8} + \sum_{s\neq 2} \frac{s(25s-15)^2}{2^{2s+5}\phi^8\big(\phi^{2s-4} - (-1)^{s}\big)} \\
&= 0.160274183422101231629872112692281197699627281556900277836798_+, \\
W_2(3) &= \frac{290^2}{2^{12}\phi^{12}} + \sum_{s\neq 3} \frac{25\cdot s(25s^2-45s+26)^2}{3\cdot 2^{2s+8}\phi^{12}\big(\phi^{2s-6} - (-1)^{s+1}\big)} \\
&= 0.120655345445838262410173716774122740053340142551806782250004_+ .+
\end{aligned}$$

In general,

$$W_2(\mathbf{X}) = 25\frac{Y_{\mathbf{X}}(\mathbf{X})^2}{(2\phi)^{4\mathbf{X}}} + \frac{25}{\mathbf{X}}\sum_{s\neq \mathbf{X}} \frac{s(Y_{\mathbf{X}}(s))^2}{2^{2s+2\mathbf{X}}(\phi^{2s+2\mathbf{X}} - (-1)^{s+\mathbf{X}}\phi^{4\mathbf{X}})},$$

where polynomials Y_s were introduced in Subsection B.3. In neither in these cases the ISC, the *Inverse Symbolic Calculator* [53] gives no results. These are new constants. Based on the identity (27), we can calculate numerical values for $W_2(n)$, and this is summarized in the Table 4.

n	$W_2(n)$	$W_2(n)\sqrt{n}$	n	$W_2(n)$	$W_2(n)\sqrt{n}$
1	0.2676322966331298	0.26763229663313	14	0.0461769229840162	0.17277822498163
2	0.1602741834221012	0.22666192389381	15	0.0443945360300747	0.17193929870708
3	0.1206553454458383	0.20898118851697	16	0.0427964974407240	0.17118598976290
4	0.0992583510259317	0.19851670205186	17	0.0413534145890492	0.17050449633061
5	0.0857703979434073	0.19178844025867	18	0.0400420336967085	0.16988396135665
6	0.0764132038812498	0.18717335912032	19	0.0388436875599233	0.16931570866817
7	0.0694718541272077	0.18380524913915	20	0.0377431970151391	0.16879270842804
8	0.0640703504588479	0.18121831712981	21	0.0367280773808485	0.16830919472793
9	0.0597172873901818	0.17915186217055	22	0.0357879546763424	0.16786038662576
10	0.0561149803515228	0.17745114876641	23	0.0349141291268267	0.16744228107545
11	0.0530716542113401	0.17601876402250	24	0.0340992439956194	0.16705149680786
12	0.0504575689489552	0.17479014609200	100	0.0157926629640446	0.15792662964044
13	0.0481814350440456	0.17372063457675	1000	0.0048047468687205	0.15193943685718

TABLE 4. The function $W_2(n)$ for $1 \leq n \leq 24$, $n = 100, 1000$.

Now we will derive asymptotic formula for the sequence $W_2(n)$.

Proposition 4. *We have an asymptotic formula*

$$W_2(n) \sim \frac{5^{1/4}}{4\sqrt{2}\sqrt{\pi}\sqrt{n}} = \frac{1}{\sqrt{n}}\cdot 0.1491394631939741_+.$$

Numerical data is presented in Table 4. However, differently from $v = 1$ case in Subsection **??**, this time, and in general for $v \geq 2$, the convergenve is only of order n^{-1}.

Proof. Let us separate (27) into four terms:

$$W_2 = K^2(\mathbf{X}, \mathbf{X}) + \sum_{i<\mathbf{X}} \frac{K(\mathbf{X}, i)K(i, \mathbf{X})}{1-(-1)^{i+\mathbf{X}}\phi^{2\mathbf{X}-2i}} + \sum_{i>\mathbf{X}} \frac{K(\mathbf{X}, i)K(i, \mathbf{X})(-1)^{i+\mathbf{X}}\phi^{2\mathbf{X}-2i}}{1-(-1)^{i+\mathbf{X}}\phi^{2\mathbf{X}-2i}}$$
$$+ \sum_{i>\mathbf{X}} K(\mathbf{X}, i)K(i, \mathbf{X}) = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4.$$

Then, minding the asymptotics of $K(i, \ell)$, we have

$$\mathcal{S}_1 = \frac{B}{\mathbf{X}}, \quad \mathcal{S}_2 = \frac{B}{\mathbf{X}}, \quad \mathcal{S}_3 = \frac{B}{\mathbf{X}},$$

and the main asymptotics comes from $\mathcal{S}_4$. □

The Table 5 summarizes numerical results.

n	$W_3(n)$	$W_3(n)\sqrt{n}$	n	$W_3(n)$	$W_3(n)\sqrt{n}$
1	0.1680289608296186	0.16802896082962	14	0.0262634917733556	0.09826898799627
2	0.0994912028536390	0.14070180841243	15	0.0252135206075704	0.09765154541237
3	0.0730151760824401	0.12646599469838	16	0.0242744501237882	0.09709780049516
4	0.0589532295056262	0.11790645901125	17	0.0234283081740980	0.09659738923134
5	0.0503513491297441	0.11258903941293	18	0.0226609289213482	0.09614217904963
6	0.0445129374472323	0.10903398369814	19	0.0219609781880675	0.09572568462311
7	0.0402451220926773	0.10647858454066	20	0.0213192670506264	0.09534266071137
8	0.0369582987959716	0.10453385479900	21	0.0207282581249678	0.09498881188226
9	0.0343299221825138	0.10298976654754	22	0.0201817037603623	0.09466058137767
10	0.0321680798925768	0.10172440041481	23	0.0196743764518228	0.09435499478917
11	0.0303507625625911	0.10066209152128	24	0.0192018649467395	0.09406954245869
12	0.0287962642532822	0.09975318550973	100	0.0087452046646542	0.08745204664654
13	0.0274475597086833	0.09896358391601	500	0.0037560671544927	0.08398821485500

TABLE 5. The function $W_3(n)$ for $1 \leq n \leq 24$, $n = 100, 500$.

5. ASYMPTOTICS FOR $W_\ell(\mathbf{X})$

In a completely analogous manner we get that

$$W_\ell(\mathbf{X}) \sim \sum_{i_1, i_2, \ldots, i_{\ell-1} > \mathbf{X}} K(\mathbf{X}, i_1) \cdot K(i_1, i_2) \cdots K(i_{\ell-1}, \mathbf{X}) \sim \frac{5^{1/4}}{2^{(\ell+1)/2}\pi^{\ell/2}\,\mathbf{X}^{1/2}} \cdot \mathscr{I}_\ell, \quad (28)$$

$$\text{where } \mathscr{I}_\ell = \int\limits_{[0,\infty)^{\ell-1}} \exp\Big(-Q_\ell(\alpha_1, \alpha_2, \ldots, \alpha_{\ell-1})\Big)\, \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_{\ell-1}.$$

Here Q_ℓ is the quadratic form

$$Q_\ell(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1})=\sum_{t=1}^{\ell-1}\alpha_t^2-\sum_{t=1}^{\ell-2}\alpha_t\alpha_{t+1}.$$

(An empty integral here is equal to 1 by convention, so the asymptotic formula holds for $\ell=1$ as well). We know, and it can be double checked, that

$$\mathscr{I}_1=1,\quad \mathscr{I}_2=\frac{\sqrt{\pi}}{2},\quad \mathscr{I}_3=\frac{2\pi}{3\sqrt{3}}.$$

Proposition 5. *For every integer $\ell\geq 2$,*

$$\mathscr{I}_\ell=\int\limits_{[0,\infty)^{\ell-1}}\exp\Bigg(-\sum_{t=1}^{\ell-1}\alpha_t^2+\sum_{t=1}^{\ell-2}\alpha_t\alpha_{t+1}\Bigg)\,d\alpha_1\cdots d\alpha_{\ell-1}=\frac{(2\pi)^{(\ell-1)/2}}{\ell^{3/2}}.$$

Proof. Let us write

$$\mathscr{I}_\ell=\int_{[0,\infty)^{\ell-1}}e^{-\mathbf{x}^TA_\ell\mathbf{x}}\,\mathrm{d}x,$$

where

$$A_\ell=\begin{pmatrix}1&-\frac12&0&\cdots&0\\-\frac12&1&-\frac12&\ddots&\vdots\\0&-\frac12&1&\ddots&0\\\vdots&\ddots&\ddots&\ddots&-\frac12\\0&\cdots&0&-\frac12&1\end{pmatrix}_{\ell-1}.$$

Since A_ℓ is positive definite,

$$\int_{\mathbb{R}^{\ell-1}}e^{-\mathbf{x}^TA_\ell\mathbf{x}}\,dx=\frac{\pi^{(\ell-1)/2}}{\sqrt{\det A_\ell}}.$$

Expanding via the bottom row, we get that

$$\det A_{\ell+1}=\det A_\ell-\frac14\det A_{\ell-1}.$$

Since $\det A_2=1$ and $\det A_3=\frac34$, this gives $\det A_\ell=\ell\cdot 2^{1-\ell}$. Therefore

$$\mathscr{K}_\ell:=\int_{\mathbb{R}^{\ell-1}}e^{-\mathbf{x}^TA_\ell\mathbf{x}}\,\mathrm{d}x=\frac{(2\pi)^{(\ell-1)/2}}{\sqrt{\ell}}.$$

Now, $2A_\ell$ is the standard tridiagonal Dirichlet Laplacian matrix. Its inverse is known, and the formula can be verified directly. Put $\Sigma=\frac12A_\ell^{-1}$. Then

$$\Sigma_{ij}=\frac{\min(i,j)(\ell-\max(i,j))}{\ell},\qquad 1\leq i,j\leq\ell-1.$$

This is exactly the covariance matrix of the discrete Gaussian bridge

$$(S_1,\ldots,S_{\ell-1})\mid S_\ell=0,$$

where $S_k=\xi_1+\cdots+\xi_k$, $\xi_1,\ldots,\xi_\ell\sim N(0,1)$ are independent. Thus the orthant probability corresponding to $\mathscr{I}_\ell$ is

$$\mathbb{P}(S_1>0,\ldots,S_{\ell-1}>0\mid S_\ell=0).$$

By the classical combinatorial Cycle Lemma, for every realization with $S_\ell = 0$ and no vanishing cyclic partial sums, exactly one of the ℓ cyclic shifts has all partial sums positive. Since the Gaussian law is continuous and cyclically invariant under the conditioning $S_\ell = 0$, this orthant probability equals ℓ^{-1}. Hence $\mathscr{I}_\ell = \ell^{-1} \cdot \mathscr{K}_\ell$. □

Gathering everything together, we thus obtain the following.

Proposition 6. *For $\ell \in \mathbb{N}$, we have the formula*

$$W_\ell(n) = \frac{5^{1/4}}{2\sqrt{\pi} \cdot \ell^{3/2}\sqrt{n}} + \frac{b(\ell, n)}{n},$$

where the function $b(\ell, n)$, for a fixed ℓ, as a function in n, is bounded from above.

It may seem from the Tables 3, 4, and 5 that $b(\ell, n) > 0$. This is not true, as we will soon see! In fact, for every fixed n there exists $L = L(n)$ such that $b(\ell, n) < 0$, for $\ell > L(n)$. Indeed, let us define now

$$T(\ell) = \sup_{n\in\mathbb{N}} |b(\ell, n)|, \quad C(\ell, n) = \frac{b(\ell, n)}{T(\ell)}.$$

Thus, $|C(\ell, n)| \leq 1$. With some "bootstrapping", we will get an improvement of Proposition 6. Indeed, by the formulas in the Subsecion **??**, we have the following:

$$\sum_{i+j=\ell} \frac{1}{(\xi_{i,j}\xi_{j,i})^{-2} - 1} < \frac{c_1}{\ell^2}, \text{ for a certain } c_1 > 0 \text{ and all } \ell \in \mathbb{N}.$$

Let $c_2 = 5^{1/4}(2\sqrt{\pi})^{-1}$. Further, minding the fact that $W_\ell(n) > 0$, by the formula (9),

$$\phi^{-4n} W_\ell(n) = \phi^{-4n} W_\ell(n) W_0(n) < \frac{c_1}{(\ell+2)^2} < \frac{c_1}{\ell^2} \Longrightarrow \left| \frac{c_2}{\ell^{3/2}\sqrt{n}} + \frac{T(\ell)C(\ell, n)}{n} \right| < \frac{c_1\phi^{4n}}{\ell^2}.$$

Imagine now that we consider only $n \in [1, \dots N]$, for some fixed $N \in \mathbb{N}$. We see that for ℓ large enough, starting from some $\ell = L$, all $C(\ell, n)$ should be negative, otherwise the above inequality and the formula (9) will be invalidated. Thus,

$$T(\ell) < \frac{1}{|C(\ell, n)|} \cdot \left| \frac{c_1\phi^{4n} n}{\ell^2} + \frac{c_2\sqrt{n}}{\ell^{3/2}} \right|, \text{ for every } n \in \mathbb{N}.$$

Setting $n = 1$, we obtain the following

$$T(\ell) < \frac{c_3}{\ell^{3/2}},$$

provided that

$$\inf_{\ell\in\mathbb{N}} |C(\ell, 1)| = c_4 > 0. \tag{29}$$

We have the following result, a refinement of (6):

Proposition 7. *For $\ell \in \mathbb{N}$, we have the formula*

$$W_\ell(n) = \frac{5^{1/4}}{2\sqrt{\pi} \cdot \ell^{3/2}\sqrt{n}} + \frac{B(\ell, n)}{\ell^{3/2} n},$$

where the function $B(\ell, n)$, is uniformly bounded.

The property (29) is equivalent too

$$\frac{c_2}{\ell^{3/2}}\sim(-b(\ell,1))>c_4|b(\ell,n)|\text{ for all }\ell,n\in\mathbb{N}.$$

This is equivalent to the inequality

$$W_\ell(n)<\frac{c_5}{\ell^{3/2}}.$$

This holds since we obtain exactly the identity (12). Similarly as in the case of $\mathcal{L}_1$, for fixed ω, $|\omega|\leq 1$, the operator $\mathcal{L}_\omega$ is of trace class and is nuclear of order zero [41]. Thus,

$$\mathrm{Tr}(\mathcal{L}_\omega)=\sum_{m=1}^{\infty}\frac{\omega^{m-1}}{\xi_m^{-2}+1},\quad \mathrm{Tr}(\mathcal{L}_\omega^2)=\sum_{i,j=1}^{\infty}\frac{\omega^{i+j-2}}{(\xi_{i,j}\xi_{j,i})^{-2}-1},$$

and similarly for higher powers, as is given by (15). Now, we know that the eigenvalues of $\mathcal{L}_\omega$ are either analytic functions of ω for $|\omega|\leq 1$, or at least can be expressed as a Puiseux series, and they are real for real ω. Let the set of eigenvalues be the union of $\{\lambda_n(\omega),n\in\mathbb{N}\}$, the set which we have constructed via (**??**), and $\{\sigma_\imath(\omega),\imath\in\mathscr{I}\}$, where $\mathscr{I}$ is a finite or a countable set. These are what we called "sporadic" eigenvalues, and it is the set which was left out of the framework of explicit construction in Section **??**. We have:

$$\sum_{n=1}^{\infty}\lambda_n^2(\omega)+\sum_{\imath\in\mathscr{I}}\sigma_\imath^2(\omega)\equiv\mathrm{Tr}(\mathcal{L}_\omega^2),\quad |\omega|\leq 1.$$

In particular, for $\omega=0$, this gives

$$\sum_{\imath\in\mathscr{I}}\sigma_\imath^2(0)=\frac{1}{\phi^4-1}-\sum_{n=1}^{\infty}\lambda_n^2(0)=0.$$

Thus, since $\sigma_\imath(0)\in\mathbb{R}$ are eigenvalues of $\mathcal{L}_0$, this implies $\mathscr{I}=\varnothing$. Also, comparing the coefficients at the powers of ω in the trace formula, we obtain

$$\sum_{n=1}^{\infty}(-1)^{n+1}\phi^{-2n}W_{\ell-1}(n)=\frac{1}{\xi_\ell^{-2}+1},\quad \ell\geq 1,$$

and similarly for higher powers, as explained in Subsection 3.3. This gives the first identities of the Theorem 2, part (ii).

6. Generating function for the coefficients $W_\ell(\mathbf{X})$

6.1. **The function $\Omega_1(w)$.** Let

$$\Omega_1(w)=\sum_{n=1}^{\infty}W_1(n)w^n=\frac{5}{4}\sum_{n=1}^{\infty}\phi^{-2n}P_{n-1}^{(0,1)}\Big(\frac{3}{2}\Big)w^n=\frac{5}{4}\sum_{n=1}^{\infty}P_{n-1}^{(0,1)}\Big(\frac{3}{2}\Big)(\phi^{-2}w)^n.$$

Then, according to the formula (35), one has

$$\begin{aligned}
\Omega_1(w) &= \frac{5w}{2(1-3\phi^{-2}w+\phi^{-4}w^2)^{1/2}(\phi^2+w)+2\phi^2(1-3\phi^{-2}w+\phi^{-4}w^2)} \\
&= \frac{5w\phi^2}{2\big((\phi^4-w)(1-w)\big)^{1/2}(\phi^2+w)+2(\phi^4-w)(1-w)} \qquad (30)\\
&= \frac{\phi^2+w}{2\big((\phi^4-w)(1-w)\big)^{1/2}}-\frac{1}{2}. \qquad (31)
\end{aligned}$$

Thus way we arrive at identities which were written down and checked numerically in Section 1:

$$-\Omega_1(-\phi^{-2})=\frac{1}{2\sqrt{2}+4},\quad \Omega_1(\phi^{-4})=\frac{1}{4\sqrt{3}+6},\quad -\Omega_1(-\phi^{-6})=\frac{1}{6\sqrt{10}+20}.$$

6.2. **The function $\Omega_{1,1}(w)$, the first way.** For $|w|<|t|<1$, $\Omega_{1,1}(w)$ is just the constant term in t of the function $\Omega_1(t)\Omega_1(\frac{w}{t})$; so,

$$\Omega_{1,1}(w)=\frac{1}{2\pi i}\oint\frac{\Omega_1(t)\Omega_1\big(\frac{w}{t}\big)}{t}\,\mathrm{d}t=\frac{25}{16}\sum_{n=1}^{\infty}\phi^{-4n}\Big(P^{(0,1)}_{n-1}(3/2)\Big)^2w^n$$

where the small contour rounds $t=0$ once in the positive direction. We have:

$$\Omega_1(t)\Omega_1\Big(\frac{w}{t}\Big)=\frac{1}{4}\left(\frac{\phi^2+t}{\big((\phi^4-t)(1-t)\big)^{1/2}}-1\right)\cdot\left(\frac{\phi^2t+w}{\big((\phi^4t-w)(t-w)\big)^{1/2}}-1\right).$$

Suppose, $0<w<1$ is real. The function $\Omega_1(t)\Omega_1(w/t)$ is a single valued function in the cut t-plane $\mathbb{C}\setminus\{[1,\phi^4]\cup[\phi^{-4}w,w]\}$ with a singular point $t=0$. Consider the contour consisting of the segment $[1-iT,1+iT]$, and a semicircle C_1, given by $1+Te^{is}$, $s\in[\frac{\pi}{2},\frac{3\pi}{2}]$. For $|t|$ large and w fixed, from (30), we have

$$\left|\frac{\Omega_1(t)\Omega_1\big(\frac{w}{t}\big)}{t}\right|\ll\frac{1}{|t|^3},$$

so the integral over C_1 is $\ll|t|^{-2}$. We now take the limit $T\to\infty$. So, $\Omega_{1,1}(w)$ is equal to the integral of $(2\pi i)^{-1}\Omega_1(t)\Omega_1(w/t)t^{-1}$ taken over the line $[1-i\infty,1+i\infty]$. By the same reasoning, this is also equal, up to the opposite sign, to the integral taken over the contour consisting of the segment $[1+iT,1-iT]$ and a semicircle C_2, given by $1+Te^{is}$, $s\in[\frac{-\pi}{2},\frac{\pi}{2}]$. Using the standard contour integration techniques, we thus obtain

$$\Omega_{1,1}(w)=\frac{1}{4\pi}\int_1^{\phi^4}\frac{\phi^2+t}{\big((\phi^4-t)(t-1)\big)^{1/2}}\cdot\left(\frac{\phi^2t+w}{\big((\phi^4t-w)(t-w)\big)^{1/2}t}-\frac{1}{t}\right)\mathrm{d}t.$$

This can be double checked with MAPLE to hold true. At the same time we obtain the identity

$$\frac{5}{4}\phi^{-2n}P^{(0,1)}_{n-1}(3/2)=\frac{1}{2\pi}\int_1^{\phi^4}\frac{\phi^2+t}{\big((t-1)(\phi^4-t)\big)^{1/2}\cdot t^{n+1}}\,\mathrm{d}t,\quad n\geq 1.$$

Thus,

$$\Omega_{1,1}(w) = \frac{1}{4\pi}\int\limits_{1}^{\phi^4} \frac{\phi^2+t}{\big((t-1)(\phi^4-t)\big)^{1/2}}\cdot\frac{(\phi^2 t+w)}{\big((t-w)(\phi^4 t-w)\big)^{1/2}t}\,\mathrm{d}t - \frac{1}{4\pi}\int\limits_{1}^{\phi^4}\frac{\phi^2+t}{\big((t-1)(\phi^4-t)\big)^{1/2}t}\,\mathrm{d}t$$

$$= \frac{1}{4\pi}\int\limits_{1}^{\phi^4} \frac{\phi^2+t}{\big((t-1)(\phi^4-t)\big)^{1/2}}\cdot\frac{\phi^2 t+w}{\big((t-w)(\phi^4 t-w)\big)^{1/2}t}\,\mathrm{d}t - \frac{1}{2}.$$

This, after some transformations, can be given the expression (using MAPLE)

$$4\pi\cdot\Big(\Omega_{1,1}(w)+\frac{1}{2}\Big)$$
$$= \frac{2(\phi^2+1)(w+\phi^2)}{\phi^4-w}\cdot K(z\sqrt{w}) + \frac{2\phi^2(1-w)}{\phi^4-w}\cdot\Pi(z,\sqrt{wz}) + \frac{2\phi^2(w-1)}{\phi^4-w}\cdot\Pi(wz,\sqrt{wz}), \quad (32)$$

where $z = \dfrac{(\phi^4-1)}{\phi^4-w}$.

According to the formula (2.1,[10]),

$$\sum_{n=0}^{\infty}\Big(P_n^{(0,1)}(3/2)\Big)^2 w^n = \frac{1}{(1+w)^2}F_4\Big(1,\frac{3}{2};1,2;\frac{1}{4(w+w^{-1}+2)},\frac{25}{4(w+w^{-1}+2)}\Big).$$

6.3. **The function $\Omega_{1,1}(w)$, the second way.** We can also present $\Omega_{1,1}(w)$ by the simpler integral using the function $\Theta(x,y)$, defined by (41):

$$\Omega_{1,1}(w) = \frac{1}{2\pi i}\oint\frac{\Theta(t,\frac{w}{t})w}{t}\,\mathrm{d}t,\quad |w|<|t|<1.$$

Thus,

$$\Omega_{1,1}(w) = \frac{5}{2\pi i}\oint\frac{w}{t(2\phi-t)(2\phi-t-\frac{w}{t}-2w\phi^{-1})}\,\mathrm{d}t$$

6.4. **The function $\Omega_2(w)$.** We already now from Subsection (3.4) that the values of $\Omega_2(w)$ at $w=(-1)^k\phi^{-2k}$ belong to $\widehat{\mathcal{P}}$.

We will demonstrate this fact in an alternative way. According to the formula (36), we have

$$K(n,i)\cdot K(i,n) = \frac{\phi^{-2n-2i}}{2^{i+n+2}(\pi i)^2}\oint\limits_{x=\frac{3}{2}}\frac{(x-1)^{i-1}(x+1)^n}{\Big(x-\frac{3}{2}\Big)^n}\,\mathrm{d}x\oint\limits_{u=\frac{3}{2}}\frac{(y-1)^{n-1}(y+1)^i}{\Big(y-\frac{3}{2}\Big)^i}\,\mathrm{d}y.$$

Thus, we need to show that the function

$$\sum_{\substack{n,i=1\\ n\neq i}}^{\infty}\frac{\phi^{-2n-2i}}{2^{i+n}(1-(-1)^{n+i}\phi^{2n-2i})}\cdot\frac{(x-1)^{i-1}(x+1)^n}{\Big(x-\frac{3}{2}\Big)^n}\cdot\frac{(y-1)^{n-1}(y+1)^i}{\Big(y-\frac{3}{2}\Big)^i}w^n$$
$$= \sum_{\substack{n,i=1\\ n\neq i}}^{\infty}\frac{\phi^{-2n-2i}}{(1-(-1)^{n+i}\phi^{2n-2i})}\cdot\frac{(x-1)^{i-1}(x+1)^n}{(2x-3)^n}\cdot\frac{(y-1)^{n-1}(y+1)^i}{(3y-3)^i}w^n$$

Let $n = i + s$, $i, s \geq 1$.

$$\frac{\phi^{-2s-4}w^{s+1}(x+1)^{s+1}(y-1)^s(y+1)}{(1-(-1)^s\phi^{2s})(2x-3)^{s+1}(2y-3)}\sum_{i=1}^{\infty}\phi^{-4i+4}\cdot\frac{(x^2-1)^{i-1}}{(2x-3)^{i-1}}\cdot\frac{(y^2-1)^{i-1}}{(2y-3)^{i-1}}\,w^{i-1}$$
$$=\frac{\phi^{-2s}w^{s+1}(x+1)^{s+1}(y-1)^s(y+1)}{\Big(1-(-1)^s\phi^{2s}\Big)(2x-3)^s\Big((2x-3)(2y-3)-(x^2-1)(y^2-1)\Big)}$$

7. Golden section and theta functions

The general Lambert-like series we obtained for $W_2(n)$ in Subsection 4.2 cannot be evaluated in closed form, but there are exceptions. For example, the 1899 result of Landau claims that ([14], p. 94)

$$\sum_{n=0}^{\infty}\frac{1}{F_{2n+1}}=\frac{\sqrt{5}}{4}\Big(\Theta_3^2(\phi^{-1})-\Theta_3^2(\phi^{-2})\Big)$$

, where the Jacobi theta function

$$\Theta_3(q)=\sum_{n\in\mathbb{Z}}q^{n^2}.$$

8. Interpolation of partial trace formulas. I

8.1. **Collapse while adding periods.** In this subsection we will see that to prove identities like (14) one does not need trace formulas. It will also be seen clearly while a certain linear combination of periods add up to algebraic numbers. Let $q=-\phi^{-2}$, $\phi^{-4}=q^2$, and write $a_{n,i}=K(n,i)K(i,n)$.

Recall that

$$A=\Omega_2(\phi^{-4})=\sum_{n=1}^{\infty}W_2(n)q^{2n},\qquad B=\Omega_{1,1}(\phi^{-4})=\sum_{n=1}^{\infty}a_{n,n}q^{2n},$$

where

$$W_2(n)=a_{n,n}+\sum_{\substack{i\geq 1\\ i\neq n}}\frac{a_{n,i}}{1-q^{i-n}}.$$

Therefore

$$2A+B=3\sum_{n=1}^{\infty}a_{n,n}q^{2n}+2\sum_{\substack{n,i\geq 1\\ i\neq n}}\frac{a_{n,i}q^{2n}}{1-q^{i-n}}.$$

Take $i>n$ and pair the terms corresponding to (n,i) and (i,n). If $i=n+d$ with $d\geq 1$, then

$$\frac{q^{2n}}{1-q^{i-n}}+\frac{q^{2i}}{1-q^{n-i}}=\frac{q^{2n}}{1-q^d}+\frac{q^{2n+2d}}{1-q^{-d}}=q^{2n}\frac{1-q^{3d}}{1-q^d}=q^{2n}+q^{n+i}+q^{2i}.$$

Hence

$$2A+B=\sum_{n=1}^{\infty}3a_{n,n}q^{2n}+\sum_{1\leq n<i}2a_{n,i}(q^{2n}+q^{n+i}+q^{2i}).$$

Since $a_{n,i}=a_{i,n}$, this may be rewritten as

$$2A+B=2\sum_{n,i\geq 1}a_{n,i}q^{2n}+\sum_{n,i\geq 1}a_{n,i}q^{n+i}.$$

The vanishing of denominators is exactly the reason for the collapse. Thus, if we define

$$T_{2,0} := \sum_{n,i\geq 1} a_{n,i} q^{2n}, \qquad T_{1,1} := \sum_{n,i\geq 1} a_{n,i} q^{n+i},$$

then we obtain the exact identity

$$\boxed{2A + B = 2T_{2,0} + T_{1,1}\,.}$$

It remains only to evaluate the two symmetric sums. One finds

$$T_{2,0} = C = \frac{2}{21+5\sqrt{21}}, \qquad T_{1,1} = D = \frac{1}{16+12\sqrt{2}}.$$

We show that

$$T_{2,0} := \sum_{n,i\geq 1} q^{2n} K(n,i)K(i,n) = \frac{2}{21+5\sqrt{21}}.$$

Using the symmetry

$$\ell\, K(j,\ell) = j\, K(\ell, j),$$

we get

$$K(n,i)K(i,n) = \frac{i}{n} K(n,i)^2.$$

Hence

$$T_{2,0} = \sum_{n,i\geq 1} q^{2n} K(n,i)K(i,n) = \sum_{n\geq 1} \frac{q^{2n}}{n} \sum_{i\geq 1} i\, K(n,i)^2.$$

For fixed n, let

$$R_n(y) := \sum_{i\geq 1} K(n,i) y^{i-1}.$$

From the defining generating function one finds

$$R_n(y) = \frac{5}{(2\phi)^{n+1}} \frac{\left(1+\frac{2y}{\phi}\right)^{n-1}}{\left(1-\frac{y}{2\phi}\right)^{n+1}}.$$

By the Hadamard-product formula,

$$\sum_{i\geq 1} i\, K(n,i)^2 = \mathrm{CT}_t \left(R_n(t)\, \frac{\mathrm{d}}{\mathrm{d}t} R_n(t^{-1}) \right);$$

(here CT_t standes for the central term, if considered as Laurent expansion in t). Therefore

$$T_{2,0} = \mathrm{CT}_t \sum_{n\geq 1} \frac{q^{2n}}{n} R_n(t)\, \frac{d}{dt} R_n(t^{-1}).$$

Since the dependence on n is geometric, the sum may be carried out explicitly, yielding

$$T_{2,0} = \mathrm{CT}_t \frac{10\big((2+\sqrt{5})t - (7+3\sqrt{5})\big)}{116t^4 + 52\sqrt{5}\,t^4 - 864t^3 - 388\sqrt{5}\,t^3 + 2121t^2 + 947\sqrt{5}\,t^2 - (1764+788\sqrt{5})t + (116+52\sqrt{5})}.$$

The only nonzero pole inside a sufficiently small circle around the origin is

$$u = \frac{33 - 11\sqrt{5} - 5\sqrt{21} + 3\sqrt{105}}{8},$$

hence $T_{2,0}$ equals the residue at $t=u$. A direct computation gives

$$T_{2,0}=\frac{2}{21+5\sqrt{21}}=C.$$

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Appendix A. Tools and preliminary results

A.1. **Jacobi polynomials.** For $\alpha,\beta\in\mathbb{Z}$, $m\in\mathbb{N}_0$, the classical Jacobi polynomials are given as follows ([49], §4.6, Formula 4.6.1 in Russian translation of the book).

$$(x-1)^{\alpha}(x+1)^{\beta}P_m^{(\alpha,\beta)}(x)=\frac{1}{2^{m+1}\pi i}\oint\limits_{\mathcal{C}}\frac{(w-1)^{m+\alpha}(w+1)^{m+\beta}}{(w-x)^{m+1}}\,\mathrm{d}w;\tag{33}$$

here a small contour $\mathcal{C}$ winds $w=x$ once in the positive direction. For arbitrary x outside the closed interval $[-1,1]$, one has an asymptotic formula ([49], Theorem 8.21.7)

$$P_m^{(\alpha,\beta)}(x)\sim\frac{\big((x+1)^{1/2}+(x-1)^{1/2}\big)^{\alpha+\beta}}{(x-1)^{\alpha/2}(x+1)^{\beta/2}(x^2-1)^{1/4}}\cdot\frac{1}{\sqrt{2\pi m}}\cdot\Big(x+(x^2-1)^{1/2}\Big)^{m+1/2},\quad\text{as } m\to\infty.\tag{34}$$

We will now need the case $m=\mathbf{X}-1$, $(\alpha,\beta)=(0,1)$ (see Theorem 2). Moreover, one can extract the second asymptotic term. In this particular case this reads as

$$\frac{5}{4}P_{\mathbf{X}-1}^{(0,1)}(3/2)\sim\frac{\phi^{2\mathbf{X}}}{\sqrt{\mathbf{X}}}\cdot\frac{5^{1/4}}{2\sqrt{\pi}}+O\left(\frac{\phi^{2\mathbf{X}}}{\mathbf{X}^{3/2}}\right),\text{ as }\mathbf{X}\to\infty$$

(see Proposition 8). Here and in the sequel one can think of $\mathbf{X}=n$, which is exactly the index of an eigenvalue λ_n - notations n or $\mathbf{X}$ are reserved for this purpose throughout this paper. Though at some places we choose to use an unspecified variable $\mathbf{X}$ to denote that it is a function, a generic version of n.

The generating function for Jacobi polynomials is crucial in our investigations. For sufficiently small w, we have ([49], formula (4.4.5)):

$$\begin{aligned}\Delta^{(\alpha,\beta)}(w)&=\sum_{m=0}^{\infty}P_m^{(\alpha,\beta)}(x)w^m\\&=2^{\alpha+\beta}(1-2xw+w^2)^{-\frac12}\\&\times\Big[1-w+(1-2xw+w^2)^{\frac12}\Big]^{-\alpha}\Big[1+w+(1-2xw+w^2)^{\frac12}\Big]^{-\beta}.\end{aligned}\tag{35}$$

In particular, when $\alpha=0$, $\beta=1$, this gives

$$\sum_{m=0}^{\infty}P_m^{(0,1)}(x)w^m=\frac{1+w}{(1-2xw+w^2)^{1/2}(x+1)w}-\frac{1}{(x+1)w}.$$

We will frequently refer to these identities. Now we can explain why precisely Jacobi polynomials $P^{(0,1)}_{m}(x)$ appear in the investigations of the operator $\mathcal{L}$.

Appendix B. The coefficients $K(j,\ell)$

Now we explore the array of algebraic numbers $K(j,\ell)\in\mathbb{Q}(\sqrt{5}\,)$, $\phi^{j+\ell}\cdot K(j,\ell)\in\mathbb{Q}_{+}$, which, in fact, is our first method (as described in the beginning of Subsection 3.3) to approach the problem. This array governs the whole collection of the eigenvalues λ_n.

B.1. **Combinatorics and asymptotics.** We will need the following crucial result. Let $\ell\in\mathbb{N}$. The jth g-coefficient of the function

$$\frac{(z+1-\phi^{-1})^{\ell-1}}{(z+1+\phi)^{\ell+1}}$$

is given by

$$K(j,\ell)=5\cdot 2^{j-\ell-2}\cdot\phi^{-\ell-j}\cdot P^{(\ell-j,1)}_{j-1}(3/2)=\frac{\phi^{-\ell-j}}{2^{j+1}\pi i}\oint\frac{(w-1)^{\ell-1}(w+1)^{j}}{\Big(w-\frac{3}{2}\Big)^{j}}\,\mathrm{d}w; \tag{36}$$

the contour winds $\frac{3}{2}$ in the positive direction. Next, we have the first and the second symmetry properties

$$\ell K(j,\ell) \;=\; jK(\ell,j),\quad j,\ell\geq 1, \tag{37}$$

$$2\phi K(j,\ell)-K(j-1,\ell) \;=\; 2\phi K(\ell,j)-K(\ell-1,j),\quad j,\ell\geq 2. \tag{38}$$

We have the "averaging" recurrence

$$2\phi K(j,\ell)=K(j-1,\ell)+2\phi^{-1}K(j-1,\ell-1)+K(j,\ell-1),\quad \ell,j\geq 2. \tag{39}$$

These coefficients are positive, and $\sum_{j=1}^{\infty}K(j,\ell)\equiv 1$ for every $\ell\in\mathbb{N}$. Further, for fixed ℓ, for $j\leq\ell$ $K(j,\ell)$ monotonically increases, achieves its maximum at $j=\ell$, and then monotonically decreases.

Proposition 8 (Asymptotics)**.** *We have*

$$K(\ell,\ell)\sim\frac{\sqrt[4]{5}}{2\sqrt{\pi}\sqrt{\ell}}.$$

(Central region). *Uniformly for $|j-\ell|<\ell^{2/3}$,*

$$K(j,\ell)=\frac{\sqrt[4]{5}}{2\sqrt{\pi}}\cdot\Big(\frac{1}{\sqrt{\ell}}+\frac{j-\ell}{4\ell^{3/2}}+\frac{B(j,\ell)}{\ell^{3/2}}\Big)\cdot\exp\Big(-\frac{\sqrt{5}(j-\ell)^2}{2(j+\ell)}\Big),$$

where $B(\ell,k)$. is bounded.
(Moderate deviations). *For $|j-\ell|>\ell^{2/3}$, we have*

$$K(j,\ell)\ll\exp\big(c\ell^{1/3}\big).$$

(Large deviations.) *For $|j-\ell|>\delta\ell$,*

$$K(j,\ell)\ll\exp\big(c_{\delta}\ell\big).$$

The window $|j-\ell| < \ell^{2/3}$ is not accidental. Saddle-point method (see Appendix B.3) gives asymptotic expansion containing terms $\frac{(j-\ell)^3}{j^2}$, $\frac{(j-\ell)^4}{j^3}$, and so on. So, these can be neglected in the given window, but the result does not automatically apply to a wider region $|j-\ell| < \ell^{3/4}$.

Proof. Next, as was shown before, the jth golden coefficient of $f(z)$ is the Taylor coefficient at x^{j-1} of the function $g(x)$; see (**??**). So, $K(j,\ell)$ is the Taylor coefficient at x^{j-1} of

$$\Delta_\ell(x) = 5 \cdot \frac{(2x\phi^{-1}+1)^{\ell-1}}{(-x+2\phi)^{\ell+1}} = \sum_{j=1}^{\infty} K(j,\ell)\cdot x^{j-1}, \tag{40}$$

and so, by a direct calculation,

$$K(j,\ell) = \frac{5\phi^{-\ell-j}}{2\pi i} \oint\limits_{|w|=1} \frac{(2w+1)^{\ell-1}}{(2-w)^{\ell+1}w^j}\,\mathrm{d}w = \frac{5}{\ell}\cdot(2\phi)^{-\ell-j}\cdot \sum_{i=0}^{\min\{\ell-1,j-1\}} \frac{(\ell+j-i-1)!}{(\ell-i-1)!(j-i-1)!}\cdot\frac{4^i}{i!}.$$

This gives the positivity (what we already know) and the first symmetry property (37). In Proposition 9, item iv), we will see how to prove this symmetry property directly using the generating function. Taking in (40) $x=1$ gives the needed summation property.

Now we will calculate the bivariate generating function of coefficients $K(j,\ell)$. According to (40), we have

$$\begin{aligned}\Theta(x,y) &= \sum_{j,\ell=1}^{\infty} K(j,\ell)x^{j-1}y^{\ell-1} \\ &= \frac{5}{(2\phi-x)^2}\sum_{\ell=1}^{\infty}\frac{(2x\phi^{-1}+1)^{\ell-1}}{(2\phi-x)^{\ell-1}}y^{\ell-1} \\ &= \frac{5}{(2\phi-x)(2\phi-x-y-2xy\phi^{-1})}. \end{aligned}\tag{41}$$

The double series converges for $|x|<1$, $|y|<1$. Explicit expression implies another symmetry property

$$\Theta(x,y)\cdot(2\phi-x) = \Theta(y,x)\cdot(2\phi-y).$$

On the level of coefficients, this reads as the second symmetry property (38).

Further, for $\ell \geq 2$, we have

$$(2\phi-x)\Delta_\ell(x) = 5\cdot\frac{(2x\phi^{-1}+1)^{\ell-1}}{(-x+2\phi)^{\ell}} = (2x\phi^{-1}+1)\Delta_{\ell-1}(x).$$

This gives the recurrence (39) with initial values

$$K(1,\ell) = \frac{5}{(2\phi)^{\ell+1}}, \qquad K(j,1) = \frac{5j}{(2\phi)^{j+1}}.$$

Let

$$\Xi(x) = \frac{2x\phi^{-1}+1}{2\phi-x} = \sum_{j=0}^{\infty} p(j)x^j.$$

Then

$$p(j) = \begin{cases} 5(2\phi)^{-j-1} \text{ for } j \geq 1, \\ (2\phi)^{-1} \text{ for } j = 0. \end{cases}$$

Let ξ be a random discrete variable with probabilities

$$\mathbb{P}(\xi = j) = p(j).$$

This is true since $\sum_{j=0}^{\infty} p(j) = \Xi(1) = 1$. Further,

$$\mathbb{E}(\xi) = \sum_{j=0}^{\infty} jp(j) = 1, \quad \mathbb{E}(\xi^2) = \sum_{j=0}^{\infty} j^2 p(j) = 1 + \frac{2}{\sqrt{5}}, \quad \mathbb{D}(\xi) = \frac{2}{\sqrt{5}}.$$

Put

$$\Delta_\ell(x) = A(x)\,\Xi(x)^{\ell-1}, \qquad A(x) = \frac{5}{(2\phi - x)^2}.$$

Khe coefficients $K(j,\ell)$ admit the probabilistic interpretation

$$k(j,\ell) = \mathbb{P}\big(U + \xi_1 + \cdots + \xi_{\ell-1} = j-1\big),$$

where U is a fixed nonnegative integer-valued random variable, and $\xi_1, \xi_2, \ldots$ are independent identically distributed nonnegative integer-valued random variables with probability generating function Ξ. In particular, ξ_1 has exponential moments in a neighborhood of the origin. Therefore the Cramér–Chernoff bound applies. It follows that there exist constants $c, C > 0$ such that

$$|j - \ell| \geq \ell^{2/3} \qquad \Longrightarrow \qquad k(j,\ell) \leq C \exp(-c\,\ell^{1/3}),$$

and, more generally, for every fixed $\delta > 0$ there exist constants $c_\delta, C_\delta > 0$ such that

$$|j - \ell| \geq \delta\ell \qquad \Longrightarrow \qquad k(j,\ell) \leq C_\delta e^{-c_\delta \ell}.$$

Thus the coefficients $k(j,\ell)$ are exponentially small outside the central strip, and in the genuinely large-deviation regime they decay exponentially in ℓ.

In the central region give results, too, but they are insufficient for our purposes. We employ the saddle-pont method instead. The needed appendix was wirtten by Nico Temme B.3.

Considering the behaviour of $K(j,\ell)$ for ℓ fixed, we can strengthen this as follows. Let $j, \ell \geq 1$.

i) Fix ℓ. $K(j,\ell)$ strictly increases until reaching its maximum value at $j = \ell$, and then strictly decreases.

ii) Fix $j \geq 2$. $K(j,\ell)$ strictly increases until reaching its maximum value at $\ell = j-1$, and then strictly decreases.

iii) $K(\ell-1,\ell) < K(\ell,\ell) < K(\ell,\ell-1) < K(\ell-1,\ell-1)$, for $\ell \geq 2$.

iv) We have:

$$1 - \frac{1}{2(\ell-1)} < \frac{K(\ell,\ell)}{K(\ell-1,\ell-1)} < 1 - \frac{1}{2\ell}.$$

$\ell\backslash j$	1	2	3	4	5	6	7	8	9	10
1	$\boxed{1}$	2	3	4	5	6	7	8	9	10
2	1	$\boxed{7}$	18	34	55	81	112	148	189	235
3	1	12	$\boxed{58}$	164	355	656	1092	1688	2469	3460
4	1	17	123	$\boxed{519}$	1530	3606	7322	13378	22599	35935
5	1	22	213	1224	$\boxed{4830}$	14556	36302	78968	155079	281410
6	1	27	328	2404	12130	$\boxed{46006}$	140532	364708	835659	1737385
7	1	32	468	4184	25930	120456	$\boxed{445012}$	1371848	3666339	8746360
8	1	37	633	6689	49355	273531	1200367	$\boxed{4352263}$	13505994	36917710
9	1	42	823	10044	86155	557106	2851597	12005328	$\boxed{42920374}$	133862060
10	1	47	1038	14374	140705	1042431	6122452	29534168	120475854	$\boxed{426019410}$

TABLE 6. Integer matrix $\{\mathbf{K}(j,\ell),\ j,\ell \in \mathbb{N}\}$. This matrix, coupled with the constant ϕ, contains all the information on eigenvalues of $\mathcal{L}$, trace formulas, and decompositions of trace formulas.

If we imagine the axis j going to the right, and the axis ℓ - downwards (as in Table 6), due to the first symmetry property, in the upper-triangle of this quadrant, the item ii) is stronger than the item i), and in the lower-triangle - the other way round. We are aware that in the proximity of the diagonal the behaviour is more complicated. Meanwhile the item iii) follows from i) and ii) immediately. Now, the item iv) follows from asymptotics. We then prove by induction on $\min\{j,\ell\}$ all i)-iii) at once using the "averaging" recurrence (39), all the cases $j=\ell-1,\ell,\ell+1$ being set by the asymptotics. For example, the inequality $K(\ell,\ell-1)>K(\ell,\ell)$ for large ℓ follows from the fact $\frac{3}{4}+B>\frac{\sqrt{5}}{4}$. We only note that the item iv) can be demonstrated to hold true purely algebraically minding the recurrence on $\mathbf{K}(n)$ of Proposition 9 below. Thus, all the above claims have a purely combinatoric-algebraic proof. $\square$

B.2. **Infinite integer matrix.** In order to understand the arithmetic structure of $K(j,\ell)$ much deeper, let us define

$$\mathbf{K}(j,\ell)=\frac{(2\phi)^{j+\ell}}{5}\cdot K(j,\ell)\in\mathbb{N}.$$

These integers for $1\le j,\ell\le 10$ are presented in Table 6. We can immediately rewrite the arithmetic part of Proposition 8.

Proposition 9. *The following identities for the integers* $\mathbf{K}(j,\ell)$ *hold:*

i) *The generating functions*

$$\sum_{j=1}^{\infty}\mathbf{K}(j,\ell)x^{j-1}=\frac{(4x+1)^{\ell-1}}{(1-x)^{\ell+1}},$$

$$\mathbf{\Theta}(x,y) \quad = \quad \sum_{j,\ell=1}^{\infty}\mathbf{K}(j,\ell)x^{j-1}y^{\ell-1}=\frac{1}{(1-x)(1-x-y-4xy)};$$

ii) *Let* $\mathbf{K}(n)=\mathbf{K}(n,n)$*; the diagonal of the rational function* $\boldsymbol{\Theta}$ *is given by*

$$\boldsymbol{\Omega}(w)=\sum_{n=1}^{\infty}\mathbf{K}(n)w^{n}=\frac{\Omega_1(4\phi^2 w)}{5}=\frac{1+4w-\sqrt{1-12w+16w^2}}{10\sqrt{1-12w+16w^2}};$$

iii) $\mathbf{K}(1,\ell)=1,\mathbf{K}(j,1)=j$*;*

iv) $\ell\mathbf{K}(j,\ell)=j\mathbf{K}(\ell,j)$*; in terms of* Θ*, this reads as*

$$\frac{\partial}{\partial y}\Big(y\Theta(x,y)\Big)=\frac{1}{(1-x-y-4xy)^2},$$

which is symmetric in (x,y)*;*

v) $\mathbf{K}(j,\ell)-\mathbf{K}(j-1,\ell)=\mathbf{K}(\ell,j)-\mathbf{K}(\ell-1,j)$*, for* $j,\ell\geq 2$*; in terms of* Θ*, this reads as*

$$(1-x)\Theta(x,y)=\frac{1}{1-x-y-4xy},$$

which is symmetric in (x,y)*;*

vi) $\mathbf{K}(j,\ell)=\mathbf{K}(j-1,\ell)+4\mathbf{K}(j-1,\ell-1)+\mathbf{K}(j,\ell-1)$*, for* $j,\ell\geq 2$*;*

vii) *For* $n\geq 3$*, we have the recurrence*

$$(n+1)\mathbf{K}(n+1)-(8n+10)\mathbf{K}(n)-(32n-72)\mathbf{K}(n-1)+64(n-2)\mathbf{K}(n-2)=0.$$

In items i) and ii) we have touched on a surface the wide research topic which investigates *diagonals* of rational functions in positive or zero characteristic.

Thus, $\boldsymbol{\Omega}$ satisfies

$$\big(5\boldsymbol{\Omega}^2+\boldsymbol{\Omega}\big)(1-12+16w^2)=w.$$

Modulo any prime $\neq 5$ this gives a quadratic equation. For example, if $F(w)=\boldsymbol{\Omega}(w)\ (\mathrm{mod}\ 2)$, then $F^2=F+w$. So,

$$F(w)=\sum_{k=1}^{\infty}w^{2^k}.$$

Since

$$\boldsymbol{\Theta}(x,y)=\frac{1}{(1-x)(1-x-y)}\ (\mathrm{mod}\ 2),$$

this gives a combinatoric corollary

$$\sum_{j=0}^{n}\binom{n+j}{j}\equiv\begin{cases}1\ (\mathrm{mod}\ 2), & \text{if } n=2^k-1,\quad k\geq 0\\ 0\ (\mathrm{mod}\ 2), & \text{if } n\neq 2^k-1,\quad k\geq 0.\end{cases}$$

On the other hand, modulo 5,

$$H(w)=\boldsymbol{\Omega}(w)\ (\mathrm{mod}\ 5)=\sum_{k=1}^{\infty}kw^k=\frac{w}{(1-w)^2}.$$

it is well-known that the item 4) of the above Theorem fails for characteristic 0, what we will soon see in Subsection 6.2 in case of the function $\Omega_{1,1}(w)$.

The sequence $\{\mathbf{K}(n), n \in \mathbb{N}\}$ is not yet contained in Online Encyclopedia of Integer Sequences [54]. However, it can be expressed in terms of the sequence A084772, which is given by

$$\frac{1}{\sqrt{1-12w+16w^2}} = \sum_{n=0}^{\infty} a(n)w^n,$$

as follows:

$$\mathbf{K}(n) = \frac{a(n)+4a(n-1)}{10}, \quad n \geq 1.$$

Finally, note that

$$\mathbf{\Omega}'(w) = \frac{1-4w}{(1-12w+16w^2)^{3/2}}.$$

This gives the identity

$$\big(10\mathbf{\Omega}(w)+1\big)\cdot(1-4w) = \mathbf{\Omega}'(w)\cdot(1+4w)\cdot\big(1-12w+16w^2\big).$$

Comparing the coefficient at w^n for $n \geq 3$ gives the item vii) of Proposition 9.
The function $\mathbf{K}(n)$, defined for natural numbers, has a natural extension as a meromorphic function

$$\mathbf{K}(z) := \frac{\Gamma(2z)}{\Gamma(z)\Gamma(z+1)}\,{}_2F_1(1-z,\,1-z;\,1-2z;\,-4) = \frac{\Gamma(2z)\cdot 5^{z-1}}{\Gamma(z)\Gamma(z+1)}\,{}_2F_1(1-z,\,-z;\,1-2z;\,4/5).$$

B.3. **$\mathbf{K}(j,\ell)$ as a function in ℓ.** We can rewrite the item v) of Proposition 9, if coupled with with the symmetry property iv), as follows:

$$(j-\ell)\mathbf{K}(j,\ell) = j\mathbf{K}(j-1,\ell) - (\ell-1)\mathbf{K}(j,\ell-1).$$

Now, let us use vi) to express $\mathbf{K}(j,\ell-1)$, and substitute this into the above expression. We obtain:

$$(j-1)\mathbf{K}(j,\ell) = (\ell+j-1)\mathbf{K}(j-1,\ell) + 4(\ell-1)\mathbf{K}(j-1,\ell-1). \tag{42}$$

Let us now introduce

$$\mathbf{K}(j,t) = Y_j(t),$$

where t is an integer, but this time we consider the latter expression as a polynomial in t. The recurrence (42) in terms of Y reads as

$$(j-1)Y_j(t) = (t+j-1)Y_{j-1}(t) + 4(t-1)Y_{j-1}(t-1), \quad Y_1(t) = 1.$$

Thus, the polynomials we obtain are as follows:

$$\begin{aligned}
Y_1(t) &= 1,\\
Y_2(t) &= 5t-3,\\
Y_3(t) &= \frac{25}{2}t^2-\frac{45}{2}t+13,\\
Y_4(t) &= \frac{125}{6}t^3-75t^2+\frac{655}{6}t-51,\\
Y_5(t) &= \frac{625}{24}t^4-\frac{625}{4}t^3+\frac{9875}{24}t^2-\frac{1925}{4}t+205,\\
Y_6(t) &= \frac{625}{24}t^5-\frac{1875}{8}t^4+\frac{23125}{24}t^3-\frac{16125}{8}t^2+\frac{25025}{12}t-819,\\
Y_7(t) &= \frac{3125}{144}t^6-\frac{4375}{16}t^5+\frac{231875}{144}t^4-\frac{83125}{16}t^3+\frac{84875}{9}t^2-\frac{35455}{4}t+3277,\\
Y_8(t) &= \frac{15625}{1008}t^7-\frac{3125}{12}t^6+\frac{149375}{72}t^5-\frac{56875}{6}t^4+\frac{3769375}{144}t^3-\frac{512225}{12}t^2+\frac{3130985}{84}t-13107.
\end{aligned}$$

Now, let

$$\sum_{j=1}^{\infty} Y_j(t)x^{j-1}=\psi(t,x).$$

We know that for $\ell\in\mathbb{N}$,

$$\psi(\ell,x)=\frac{(4x+1)^{\ell-1}}{(1-x)^{\ell+1}}.$$

And in general, for $t\in\mathbb{R}$,

$$\psi(t,x)=\frac{(4x+1)^{t-1}}{(1-x)^{t+1}}.$$

Indeed, the initial condition is $\psi(t,0)=1$, and we note that the Taylor coefficients of the latter function satisfy the same recurrence, minding the differential equation

$$(1-x)\frac{\partial}{\partial x}\,\psi(t,x)=(t+1)\psi(t,x)+4(t-1)\psi(t-1,x).$$

Thus,

$$Y_j(t)=\sum_{s=0}^{j-1}\binom{t-1}{s}4^s(-1)^{j+s+1}\binom{-t-1}{j-s-1}.$$

We will need these polynomials further, starting from Subsection 4.2.

Appendix C. Asymptotics of Jacobi polynomials, in collaboration with Nico Temme

Consider the integral

$$Q(j,\ell)=\frac{1}{2\pi i}\oint\frac{(w-1)^{\ell-1}(w+1)^j}{(w-\frac{3}{2})^j}\,\mathrm{d}w,$$

where the contour winds $\frac{3}{2}$ in the positive direction. Wer will derive its uniform asymptotics uniformly for

$$|j-\ell|<\ell^{2/3}. \tag{43}$$

We write this in the form

$$Q(j,\ell) = \frac{1}{2\pi i}\oint e^{\ell f(w)}\frac{\mathrm{d}w}{w-1}, \tag{44}$$

where

$$f(w) = \ln(w-1) + \mu\ln(w+1) - \mu\ln\left(w-\frac{3}{2}\right), \quad \mu = \frac{j}{\ell}.$$

We have

$$f'(w) = \frac{2w^2 - w - 3 - 5\mu w + 5\mu}{(w-1)(w+1)(2w-3)}.$$

The saddle points are

$$w_\pm(\mu) = \frac{1}{4}\left(1 + 5\mu \pm W\right), \quad W = \sqrt{25 - 30\mu + 25\mu^2}. \tag{45}$$

The argument of the square root W is positive for all $\mu \geq 0$. So, the saddle points are real, and w_+ is monotonously increasing for $\mu \in (0,\infty)$, starting at the value $\mu_+(0) = \frac{3}{2}$. It turns out after some analysis that w_+ is the dominant saddle point. It should be remarked that, because j and ℓ are integers, the integrand in (44) is single-valued, but that the function $f(w)$ should be described in terms of its branch cuts. Because contributions to the integral from parts of the contour of w with $\Re(w) < \frac{3}{2}$ are exponentially small compared with those from the neighbourhood of the saddle point w_+, we forget about this aspect.

Now, note that

$$f''(w) + (f'(w))^2 = \frac{5(5\mu w - 3w - 5\mu + 7)\mu}{(2w-3)^2(w+1)^2(w-1)}.$$

For the large-ℓ asymptotics we need

$$f''(w_+) = \frac{1}{40\mu}W\Big(17 - 30\mu + 25\mu^2 + (3-5\mu)W\Big).$$

The first order asymptotic result follows from replacing $f(w)$ in (44) by $f(w_+) + \frac{1}{2}f''(w_+)(w-w_+)^2$, which gives when we integrate along the vertical line $\Re w = w_+$

$$Q(j,\ell) = \frac{e^{\ell f(w_+)}}{(w_+ - 1)\sqrt{2\pi f''(w_+)}}\Big(1 + B\ell^{-1}\Big), \text{ as } \ell\to\infty.$$

Next,

$$\frac{\mathrm{d}}{\mathrm{d}\mu} f\Big(w_+(\mu),\mu\Big) = \frac{\partial}{\partial w} f(w_+,\mu)\cdot w_+' + \frac{\partial}{\partial \mu} f(w_+,\mu).$$

If $\ell=j$, or $\mu=1$, we have

$$\begin{aligned} w_{+}(1) &= \phi^2,\\ w'_{+}(\mu)\big|_{\mu=1} &= \frac{5+\sqrt{5}}{4},\\ \frac{\partial}{\partial w}f(w,\mu)\big|_{\mu=1} &= 0.\\ \frac{\partial}{\partial \mu}f(w,\mu)\big|_{\mu=1} &= \ln(2\phi),\\ f(w,\mu)\big|_{\mu=1} &= \ln(2\phi^2). \end{aligned}$$

Thus,

$$f(w_{+}(\mu),\mu)\Big|_{\mu=1}=\phi^2,$$

$$\frac{\mathrm{d}}{\mathrm{d}\mu}f(w_{+}(\mu),\mu)\Big|_{\mu=1}=\ln(2\phi).$$

This the same manner we calculate

$$\frac{\mathrm{d}^2}{\mathrm{d}\mu^2}f(w_{+}(\mu),\mu)\Big|_{\mu=1}=-\frac{\sqrt{5}}{4}.$$

$$f(w_{+}(\mu),\mu) \quad = \quad \ln(2\phi^2)+\ln(2\phi)(\mu-1)-\frac{\sqrt{5}}{2}(\mu-1)^2+B(\mu-1)^3.$$

Let

$$g(\mu)=f(w_{+}(\mu),\mu)-\ln(2\phi^2)-\ln(2\phi)(\mu-1)=$$

$$e^{\ell f(w_{+})}=(2\phi^2)^{\ell}\cdot(2\phi)^{j-\ell}\exp\Big(-\frac{(j-\ell)^2}{4\ell}\Big)\exp\Big(\frac{B(j-\ell)^3}{\ell^2}\Big).$$

Further,

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\frac{1}{w_{+}(\mu)-1}=-\frac{w'_{+}(\mu)}{(w_{+}(\mu)-1)^2}.$$

$$\frac{1}{w_{+}-1}=\phi^{-1}+\frac{\sqrt{5}\phi^{-1}}{4}(\mu-1)+B(\mu-1)^2.$$

Finally,

$$\frac{1}{\sqrt{f''(w_{+})}}=\frac{\sqrt[4]{5}\phi}{\sqrt{2}}+(\mu-1)+B(\mu-1)^2.$$

Because

$$f''(w_{+})=\frac{1}{\mu}(4-\tfrac{52}{5}\mu+\mathcal{O}(\mu^2)),\quad \mu\to 0,$$

and

$$f''(w_{+})=\frac{1}{\mu^2}\left(\tfrac{4}{25}+\tfrac{12}{125}\mu^{-1}+\mathcal{O}\left(\mu^{-2}\right)\right),\quad \mu\to\infty,$$

We give the first coefficients of the expansion

$$Q(\ell,j) \sim \frac{(w_+-1)^{\ell-1}(w_++1)^j}{\left(w_+-\frac{3}{2}\right)^j}\frac{1}{\sqrt{2\pi\ell f''(w_+)}}\sum_{k=0}^{\infty}\frac{a_k(\mu)}{\ell^k}, \quad \ell\to\infty. \tag{46}$$

First we introduce a transformation

$$f(w)-f(w_+)=\tfrac{1}{2}s^2,$$

where w runs along the saddle point contour through w_+ and s on the imaginairy axis. We use this transformation in (44) and obtain

$$Q(\ell,j)=\frac{e^{\ell f''(w_+)}}{2\pi i}\int_{-i\infty}^{i\infty}e^{\frac{1}{2}\ell s^2}g(s)\,\mathrm{d}s, \quad g(s)=\frac{1}{w-1}\frac{dw}{ds}. \tag{47}$$

We need the coefficients of the expansion

$$w=w_0+\sum_{k=1}^{\infty}c_ks^k, \quad c_1=\left.\frac{dw}{ds}\right|_{s=0}=\frac{1}{\sqrt{f''(w_+)}},$$

where we take the positive square root; in this way, w crosses the saddle point w_+ in the same direction as s crosses the origin. The next one is

$$c_2=\frac{(w_+-1)(3w_+^2-3w_+-1-c_1^2)}{3(2-2w_++w_+^2)}.$$

With these coefficients we expand $g(s)$ defined in (47), $g(s)=\sum_{k=0}^{\infty}g_ks^k$. Then the coefficients $a_k(\mu)$ of (46) are given by

$$a_k(\mu)=\frac{(-1)^k2^k\left(\frac{1}{2}\right)_k\,g_{2k}}{g_0}, \quad g_0=\frac{c_1}{w_+-1}, \quad k=0,1,2,,\ldots.$$

The first few are $a_0(\mu)=1$ and, with W defined in (45),

$$\begin{aligned}
a_1(\mu) &= -\frac{(5\mu^2+2\mu+5)(5\mu^2-11\mu+5)W}{60\mu(5-6\mu+5\mu^2)^2},\\
a_2(\mu) &= \frac{1}{288\mu^2(5-6\mu+5\mu^2)^3}\big(125\mu^8-450\mu^7+685\mu^6-\\
&\quad 4410\mu^5+9972\mu^4-4410\mu^3+685\mu^2-450\mu+125\big),\\
a_3(\mu) &= \frac{W}{51840(\mu^3(5-6\mu+5\mu^2)^5}\big(86875-469125\mu+\\
&\quad 1384725\mu^2-2389500\mu^3+2957775\mu^4-12518415\mu^5+\\
&\quad 23051362\mu^6-12518415\mu^7+2957775\mu^8-2389500\mu^9+\\
&\quad 1384725\mu^{10}-469125\mu^{11}+86875\mu^{12}\big).
\end{aligned}$$

These coefficients are bounded as $\mu\to\infty$, but become undefined as $\mu\to0$, that is $j/\ell\to0$. For the expansion in (46) we need the condition $\mu\ge\mu_0>0$, where μ_0 is fixed number.

Appendix D. Proof af asymptotics

Proposition 10. *Fix $\ell \geq 1$. Assume that the coefficients $q_v^{(j)}(n)$ and $Y_v(n)$ satisfy*

$$q_v^{(j)}(n)\Big(1-\frac{\lambda_j}{\lambda_n}\Big) = \sum_{i=1}^{\infty} q_{v-1}^{(i)}(n)\, k(j,i) - \frac{\lambda_j}{\lambda_n}\sum_{r=1}^{v} Y_r(n)\, q_{v-r}^{(j)}(n), \qquad v \geq 1, \tag{48}$$

with

$$Y_v(n) = \sum_{i=1}^{\infty} q_{v-1}^{(i)}(n)\, k(n,i), \qquad q_0^{(j)}(n) = \delta_{j,n}, \qquad q_v^{(n)}(n) = 0 \ \ (v \geq 1), \tag{49}$$

and

$$\lambda_j = (-1)^{j+1}\phi^{-2j}.$$

Assume further that the kernel $k(j,i)$ satisfies all estimates needed below, namely:

(a) *every* pure upper chain *of length m is of size $O(c_m n^{-1/2})$;*

(b) *every chain of length m containing at least one* defect *is of size $O(d_m n^{-1})$;*

(c) *the sequences $(c_m)_{m\geq 1}$ and $(d_m)_{m\geq 1}$ are summable.*

Here a defect means one of the following:

- *at least one intermediate index is $\leq n$;*
- *at least one factor $\Big(1-\dfrac{\lambda_j}{\lambda_n}\Big)^{-1} - 1$ with $j > n$ occurs;*
- *at least one factor $\dfrac{\lambda_j}{\lambda_n}$ occurs.*

Then there exists a summable sequence $C_\ell \geq 0$ such that

$$Y_\ell(n) = \sum_{i_1,\dots,i_{\ell-1}>n} k(n,i_1)\, k(i_1,i_2)\cdots k(i_{\ell-1},n) + O\Big(\frac{C_\ell}{n}\Big).$$

Proof. Put

$$\mu_{j,n} := \frac{\lambda_j}{\lambda_n} = (-1)^{j-n}\phi^{2n-2j}.$$

Then (48) becomes

$$q_v^{(j)}(n)\big(1-\mu_{j,n}\big) = \sum_{i=1}^{\infty} q_{v-1}^{(i)}(n)\, k(j,i) - \mu_{j,n}\sum_{r=1}^{v} Y_r(n)\, q_{v-r}^{(j)}(n). \tag{50}$$

We first isolate the principal part of the recurrence. For $j > n$,

$$\mu_{j,n} = (-1)^{j-n}\phi^{-2(j-n)},$$

hence $\mu_{j,n}$ is exponentially small in $j-n$, and

$$\frac{1}{1-\mu_{j,n}} = 1 + \Big(\frac{1}{1-\mu_{j,n}} - 1\Big),$$

where the correction term is again exponentially small in $j-n$. For $j < n$, one has

$$\frac{1}{1-\mu_{j,n}} = O\big(\phi^{-2(n-j)}\big),$$

so going below n brings a strong penalty.

Motivated by this, define the *principal coefficients* $p_v^{(j)}(n)$ by

$$p_0^{(j)}(n) = \delta_{j,n},$$

and for $v \geq 1$,

$$p_v^{(j)}(n) = \begin{cases} \sum_{i>n} p_{v-1}^{(i)}(n)\, k(j,i), & j > n, \\ 0, & j \leq n. \end{cases}$$

An easy induction gives

$$p_v^{(j)}(n) = \sum_{i_1,\dots,i_{v-1}>n} k(j,i_1)\, k(i_1,i_2) \cdots k(i_{v-1},n), \qquad j > n. \tag{51}$$

In particular,

$$\sum_{j>n} k(n,j)\, p_{\ell-1}^{(j)}(n) = \sum_{i_1,\dots,i_{\ell-1}>n} k(n,i_1)\, k(i_1,i_2) \cdots k(i_{\ell-1},n). \tag{52}$$

We claim that for each fixed $v \geq 1$ there exists a summable sequence $E_v(\cdot)$ such that

$$\sum_{j>n} k(n,j) \left| q_v^{(j)}(n) - p_v^{(j)}(n) \right| + \sum_{j<n} k(n,j) \left| q_v^{(j)}(n) \right| = O\!\left(\frac{E_v}{n} \right). \tag{53}$$

Once this is proved for $v = \ell - 1$, (49) gives

$$Y_\ell(n) = \sum_{j\geq 1} q_{\ell-1}^{(j)}(n)\, k(n,j) = \sum_{j>n} p_{\ell-1}^{(j)}(n)\, k(n,j) + O\!\left(\frac{E_{\ell-1}}{n} \right),$$

and then (52) yields the required formula with $C_\ell = E_{\ell-1}$.

It remains to prove (53). We proceed by induction on v.

Base step: $v = 1$. From (50), for $j \neq n$,

$$q_1^{(j)}(n) = \frac{k(j,n)}{1 - \mu_{j,n}}.$$

If $j > n$, then

$$q_1^{(j)}(n) = k(j,n) + \left(\left(1 - \mu_{j,n} \right)^{-1} - 1 \right) k(j,n).$$

Since $p_1^{(j)}(n) = k(j,n)$ for $j > n$, the difference $q_1^{(j)}(n) - p_1^{(j)}(n)$ is exactly a defective chain of length 1, hence contributes $O(d_1 n^{-1})$ after weighting by $k(n,j)$ and summing over $j > n$. If $j < n$, then

$$q_1^{(j)}(n) = \frac{k(j,n)}{1 - \mu_{j,n}},$$

and this is again a defective chain of length 1 by definition. Thus (53) holds for $v = 1$.

Induction step. Assume (53) holds for all levels $< v$. We prove it for v.

Take first $j > n$. Solving (50) for $q_v^{(j)}(n)$, we get

$$q_v^{(j)}(n) = \frac{1}{1 - \mu_{j,n}} \sum_{i\geq 1} q_{v-1}^{(i)}(n)\, k(j,i) - \frac{\mu_{j,n}}{1 - \mu_{j,n}} \sum_{r=1}^{v} Y_r(n)\, q_{v-r}^{(j)}(n). \tag{54}$$

Now decompose the first factor as

$$\frac{1}{1-\mu_{j,n}} = 1 + \left(\frac{1}{1-\mu_{j,n}} - 1\right),$$

and split the sum over i into $i > n$ and $i < n$. This yields

$$q_v^{(j)}(n) = A_v^{(j)}(n) + R_v^{(j)}(n),$$

where

$$A_v^{(j)}(n) := \sum_{i>n} q_{v-1}^{(i)}(n)\, k(j,i),$$

and every term in $R_v^{(j)}(n)$ contains at least one defect: either an index $i < n$, or a factor $\big((1-\mu_{j,n})^{-1} - 1\big)$, or a factor $\mu_{j,n}$.

Next, write

$$A_v^{(j)}(n) = \sum_{i>n} p_{v-1}^{(i)}(n)\, k(j,i) + \sum_{i>n} \big(q_{v-1}^{(i)}(n) - p_{v-1}^{(i)}(n)\big) k(j,i).$$

The first sum is exactly $p_v^{(j)}(n)$ by definition. Thus

$$q_v^{(j)}(n) - p_v^{(j)}(n) = \sum_{i>n} \big(q_{v-1}^{(i)}(n) - p_{v-1}^{(i)}(n)\big) k(j,i) + R_v^{(j)}(n).$$

Multiply by $k(n,j)$ and sum over $j > n$. The first term is controlled by the induction hypothesis applied at level $v-1$, together with the pure-chain bounds. The remainder $R_v^{(j)}(n)$ is a sum of defective chains of total length v, and is therefore $O(d_v n^{-1})$ by assumption. Hence

$$\sum_{j>n} k(n,j)\, \big|q_v^{(j)}(n) - p_v^{(j)}(n)\big| = O\left(\frac{\widetilde{E}_v}{n}\right)$$

for some sequence $\widetilde{E}_v$ obtained from finitely many linear combinations and convolutions of the sequences d_m and the previously constructed E_u.

Now take $j < n$. Formula (54) still holds, but now the prefactor $(1-\mu_{j,n})^{-1}$ itself is defective. Therefore every term in the full expansion of $q_v^{(j)}(n)$ is a defective chain of total length v, and thus

$$\sum_{j<n} k(n,j)\, \big|q_v^{(j)}(n)\big| = O\left(\frac{\widehat{E}_v}{n}\right).$$

Combining the two parts proves (53) at level v.

Finally, observe that each E_v is obtained from finitely many sums and convolutions of the summable sequences (c_m) and (d_m). Since the convolution of summable sequences is summable, the sequence (E_v) is summable as well. This completes the proof. □

Lemma (pure upper-chain bound)**.** *For every $m \geq 1$,*

$$\sum_{i_1,\ldots,i_m>n} k(n,i_1)\, k(i_1,i_2) \cdots k(i_m,n) = O(c_m n^{-1/2}),$$

with $\sum_{m\geq 1} c_m < \infty$.

Lemma (defective chain bound)**.** *Every chain of length m containing at least one defect is*

$$O(d_m n^{-1}),$$

with $\sum_{m\geq 1} d_m < \infty$.

Lemma (bound for Y_r)**.** *For every fixed $r \geq 1$,*

$$Y_r(n) = O(c_r n^{-1/2}),$$

with the same summable sequence (c_r) as above.

REFERENCES

[1] G. ALKAUSKAS, Transfer operator for the Gauss' continued fraction map. II. Fine arithmetic of the decomposition formulas (*in preparation*).

[2] G. ALKAUSKAS, Transfer operator for the Gauss' continued fraction map. III. Selberg zeta function and structure of the eigenvalues of the Mayer-Ruelle operator (*in preparation*).

[3] G. ALKAUSKAS, The moments of Minkowski question mark function: the dyadic period function, *Glasg. Math. J.* 52 (1) (2010), 41–64.

[4] G. ALKAUSKAS, Generating and zeta functions, structure, spectral and analytic properties of the moments of the Minkowski question mark function, *Involve* **2** (2) (2009), 121-159.

[5] G. ALKAUSKAS, The Minkowski question mark function: explicit series for the dyadic period function and moments, *Math. Comp.* **79** (269) (2010), 383–418. Addenda and corrigenda, *Math. Comp.* **80** (276) (2011), 2445–2454.

[6] G. ALKAUSKAS, Recursive construction of a series converging to the eigenvalues of the Gauss-Kuzmin-Wirsing operator, `arXiv:1004.1783`.

[7] A. I. APTEKAREV, V. S. BUYAROV, On the resolvent of the Gauss operator. Russian version appears in *Mat. Zametki* **94**(4) (2013), 628–631. *Math. Notes* **94**(3-4) (2013), 590–593.

[8] K. I. BABENKO, A problem by Gauss, *Dokl. Akad. Nauk SSSR* **238** (5) (1978), 1021–1204; English translation: *Soviet Math. Dokl.* **19** (1) (1978), 136–140.

[9] K. I. BABENKO, S. P. YUR'EV, On a problem of Gauss, *Selecta Math. Soviet.* **2**(4) (1982), 331–378.

[10] W. N. BAILEY, The generating function of Jacobi polynomials, *J. London Math. Soc.* **13** (1938), 8–12.

[11] F. BEUKERS, Hypergeometric functions, how special are they? *Notices Amer. Math. Soc.* **61** (1) (2014), 48—56.

[12] K. BRIGGS (2003), A precise computation of the Gauss-Kuzmin-Wirsing constant, `http://keithbriggs.info/documents/wirsing.pdf`.

[13] P.S. BRUCKMAN, On the evaluation of certain infinite series by elliptic functions. *Fibonacci Quart.* **15**(4) (1977), 293–310.

[14] J. M. BORWEIN, JONATHAN M, P. B. BORWEIN, *Pi and the AGM. A study in analytic number theory and computational complexity.* Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1987.

[15] CH.-H. CHANG, D. MAYER, The transfer operator approach to Selberg's zeta function and modular and Maass wave forms for $\mathrm{PSL}(2,\mathbf{Z})$. In: *Emerging applications of number theory (Minneapolis, MN, 1996)*, 73–141, IMA Vol. Math. Appl., 109, Springer, New York, 1999.

[16] A. DURNER, On a theorem of Gauss-Kuzmin-Lévy, *Arch. Math. (Basel)* **58**(3) (1992), 251–256.

[17] H. DAUDÉ, PH. FLAJOLET, B. VALLÉE, An average-case analysis of the Gaussian algorithm for lattice reduction, *Combin. Probab. Comput.* **6** (4) (1997), 397–433.

[18] S.R. FINCH, *Mathematical constants*, Encyclopedia of Mathematics and its Applications, 94. Cambridge University Press, Cambridge (2003).

[19] PH. FLAJOLET, B. VALLÉE (1995), On the Gauss-Kuzmin-Wirsing constant, `http://algo.inria.fr/flajolet/Publications/gauss-kuzmin.ps`.

[20] PH. FLAJOLET, B. VALLÉE, Continued fraction algorithms, functional operators, and structure constants, *Theoret. Comput. Sci.* **194** (1-2) (1998), 1–34.

[21] PH. FLAJOLET, B. VALLÉE, Continued fractions, comparison algorithms, and fine structure constants, *Constructive, experimental, and nonlinear analysis (Limoges, 1999)*, 53–82, CMS Conf. Proc., 27, Amer. Math. Soc., Providence, RI, 2000.

[22] H. FURSTENBERG, Algebraic functions over finite fields, *J. Algebra* **7** (1967) 271–277.

[23] R. GARUNKŠTIS, J. STEUDING, On the distribution of zeros of the Hurwitz zeta-function. *Math. Comp.* **76**(257) (2007), 323–337.

[24] R. GARUNKŠTIS, J. STEUDING, Questions around the nontrivial zeros of the Riemann zeta-function. Computations and classifications. *Math. Model. Anal.* **16**(1) (2011), 72–81.

[25] B. V. GNEDENKO, On the local limit theorem in the theory of probability, *Uspekhi Mat. Nauk*, **3**, 1948, 187—194.

[26] D. HENSLEY, The number of steps in the Euclidean algorithm, *J. Number Theory* **49**(2) (1994), 142–182.

[27] D. HENSLEY, *Continued fractions*, World Scientific Publishing Co. Pte. Ltd. (2006).

[28] D. HENSLEY, Continued fractions, Cantor sets, Hausdorff dimension, and transfer operators and their analytic extension, *Discrete Contin. Dyn. Syst.* **32**(7) (2012), 2417—2436.

[29] A.F. HORADAM, Elliptic functions and Lambert series in the summation of reciprocals in certain recurrence-generated sequences. *Fibonacci Quart.* **26**(2) (1988), 98–114.

[30] M. IOSIFESCU, Spectral analysis for the Gauss problem on continued fractions, *Indag. Math. (N.S.)* **25**(4) (2014), 825–831.

[31] M. IOSIFESCU, On the Gauss-Kuzmin-Lévy theorem, *I. Rev. Roumaine Math. Pures Appl.* **39**(2) (1994), 97–117.

[32] H. IWANIEC, *Spectral methods of automorphic forms.* Second edition. Graduate Studies in Mathematics, 53. American Mathematical Society, Providence, RI; Revista Matemática Iberoamericana, Madrid, 2002.

[33] O. JENKINSON, L. F. GONZALEZ M. URBAŃSKI, On transfer operators for continued fractions with restricted digits, PROC. LONDON MATH. SOC.(3) **86**(3) (2003, 755–778.

[34] A. YA. KHINCHIN, *Continued fractions*, The University of Chicago Press, 1964.

[35] D.E. KNUTH, *The art of computer programming*, 2nd ed., vol 2: Seminumerical algorithms, Addison-Wesley, 1981.

[36] M. KONTSEVICH, D. ZAGIER, Periods, *Mathematics unlimited - 2001 and beyond*, 771-808, Springer, Berlin, 2001.

[37] J. LEWIS, D. ZAGIER, Period functions and the Selberg zeta function for the modular group, *The mathematical beauty of physics (Saclay, 1996)*, 83–97, Adv. Ser. Math. Phys., 24, *World Sci. Publ., River Edge, NJ*, 1997.

[38] L. LIPSHITZ, A. J. VAN DER POORTEN, Rational functions, diagonals, automata and arithmetic, *Number theory (Banff, AB, 1988)*, 339–358, de Gruyter, Berlin, 1990.

[39] J. LEWIS, D. ZAGIER, Period functions for Maass wave forms. I, *Ann. of Math. (2)* **153** (2001), no. 1, 191–258.

[40] A. J. MACLEOD, High-accuracy numerical values in the Gauss-Kuz'min continued fraction problem, *Comput. Math. Appl.* **26** (3) (1993), 37–44.

[41] D. MAYER, On a ζ function related to the continued fraction transformation, *Bull. Soc. Math. France* **104** (2) (1976), 195–203.

[42] D. MAYER, G. ROEPSTORFF, On the relaxation time of Gauss's continued-fraction map. I. The Hilbert space approach (Koopmanism), *J. Statist. Phys.* **47** (1-2) (1987), 149–171.

[43] D. MAYER, G. ROEPSTORFF, On the relaxation time of Gauss' continued-fraction map. II. The Banach space approach (transfer operator method), *J. Statist. Phys.* **50** (1-2) (1988), 331–344.

[44] D. MAYER, The thermodynamic formalism approach to Selberg's zeta function for $\mathrm{PSL}(2,\mathbb{Z})$, *Bull. Amer. Math. Soc. (N.S.)* **25** (1) (1991), 55–60.

[45] D. MAYER, On the thermodynamic formalism for the Gauss map, *Comm. Math. Phys.* **130** (2) (1990), 311–333.

[46] D. MAYER, Continued fractions and related transformations. In: *Ergodic theory, symbolic dynamics, and hyperbolic spaces* (Trieste, 1989), Oxford Sci. Publ., Oxford Univ. Press, New York, 1991 (175–222).

[47] J.M. RIBANDO, Measuring solid angles beyond dimension three, *Discrete Comput. Geom.* **36** (3) (2006), 479–487.

[48] P. SEBAH, Computing eigenvalues of Wirsing's operator (2012); `http://mif.vu.lt/~alkauskas/MP3/eigen-sebah.txt`.

[49] G. SZEGŐ, *Orthogonal polynomials.* Fourth edition. American Mathematical Society, Colloquium Publications, Vol. XXIII. American Mathematical Society, Providence, R.I. (1975). Russian translation: 1962, Moscow. Formulas and paragraph numbers cited in the paper refer to the Russian translation.

[50] N. M. Temme, *Asymptotic methods for integrals.* Series in Analysis, 6. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015.

[51] E. WIRSING, On the theorem of Gauss-Kusmin-Lévy and a Frobenius-type theorem for function spaces, *Acta Arith.* **24** (1973/74), 507–528.

[52] D. ZAGIER (2001), New points of view on the Selberg zeta function, Proceedings of the Japanese-German Seminar "Explicit Structures of Modular Forms and Zeta Functions", Ryushi-do (2002) http://people.mpim-bonn.mpg.de/zagier/files/tex/NewPointsSelbergZeta/fulltext.pdf.

[53] The Inverse Symbolic Calculator, available at: https://isc.carma.newcastle.edu.au/.

[54] The On-line Encyclopedia of Integer Sequences, https://oeis.org/, sequence A084772.

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