FRIENDLY PATHS FOR FINITE SUBSETS OF PLANE INTEGER LATTICE

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Abstract. For a given finite subset $\mathcal{P}$ of points of the lattice $\mathbb{Z}^2$, *a friendly path* is a monotone (uphill or downhill) lattice path (compound of vertical and horizontal segments) which splits points in half; points lying on the path itself are discarded. The purpose of this note is to describe all possible configurations of n points in $\mathbb{Z}^2$ which do not admit a friendly line. We call such configuration *inseparable*. Up to lattice symmetry, this can be done in exactly $c(n)$ ways, where the latter is the new entry into OEIS. In particular, $n = 27$ and $n = 43$ are the first odd numbers with $c(n) = 1$, while $n = 45$ is the first odd number with $c(n) = 2$. This solves the problem 11484(b)* posed in American Mathematical Monthly (February 2010).

1. Friendly paths

The following problem appeared in American Mathematical Monthly ([1], Problem 11484).

Problem. *An* uphill *lattice path is the union of a (doubly infinite) sequence of directed line segments in* $\mathbb{R}^2$, *each connecting an integer pair* (a, b) *to an adjacent pair, either* $(a, b+1)$ *or* $(a+1, b)$. *A* downhill *lattice path is defined similarly, but with* $b-1$ *in place of* $b+1$, *and a* monotone *lattice path is an uphill or downhill lattice path.*

Given a finite set $\mathcal{P}$ *of points in* $\mathbb{Z}^2$, *a* friendly *path is a monotone lattice path for which there are as many points in* $\mathcal{P}$ *on one side of the path as on the other (points that lie on the path do not count).*

- (**a**) *Prove that if* $N = b^2 + a^2 + b + a$ *for some positive integers* a, b *such that* $a \leq b \leq a + \sqrt{2a}$, *there exists a configuration of* N *such points that there does not exist a friendly line.*
- (**b**)* *Is it true that for every odd-sized set of points there is a friendly path?*

Asterisk means that no solution was known to the presenter (myself). None was received by the editors [2]. In this note we completely solve this problem. Recall that plane integer lattice symmetry group is $\mathbb{D}_4 \ltimes \mathbb{Z}^2$, denoted in crystallography by $p4m$.

Proposition 1. *Suppose* $n \in \mathbb{N}$ *is odd,* $n \leq 41$, *and a set of* n *lattice points does not admit a friendly line. Then* $n = 27$ *and the unique configuration (up to lattice symmetry group), is presented inn Figure 1.*

Here are first 45 members of the sequence $c(n)$ (offset is $n = 1$):

0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 2, 0, 1, 0, 0, 0, 1, 0, 3, 1, ., 0, ., 0, ., 0, ., 0, ., 0, ., 0, ., 0, ., 1, 6, 2.

Since only monotone paths will be considered, the adjective "monotone" will be skipped altogether. Paths will be donoted by curly letters $\mathscr{A}, \mathscr{B}$, and so on. A finite lattice subset without a

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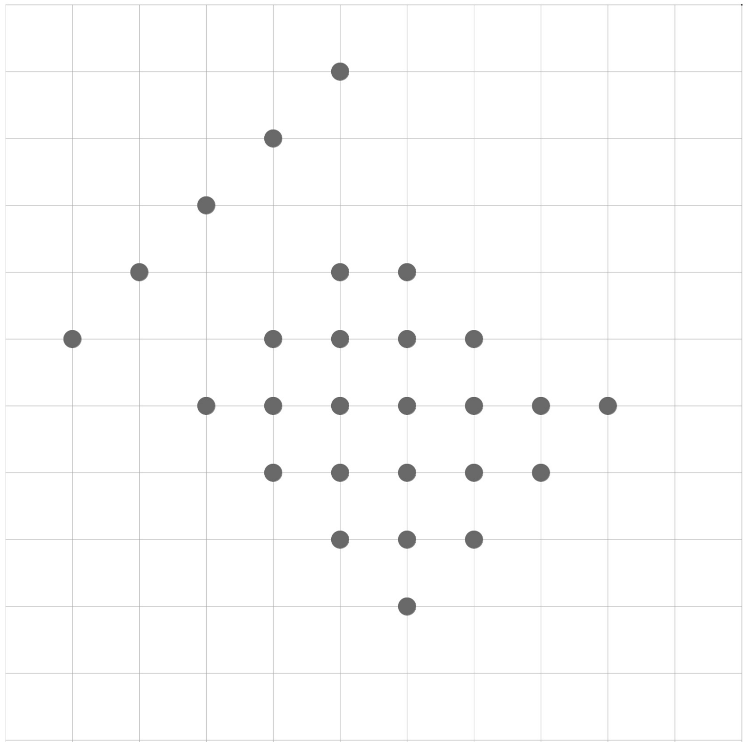

FIGURE 1. The unique odd-sized set with ≤ 41 points having no friendly path

friendly path will be called *inseparable*.

For a path $\mathscr{A}$, let $d(\mathscr{A}) = \ell(\mathscr{A}) - r(\mathscr{A})$, where ℓ and r are number of points from $\mathcal{P}$ on the left and, respectively, on the right shores of $\mathscr{A}$. We will call this number *the balance* of the path $\mathscr{A}$. A *turn* on the path can be either left or right. Let us gather all right and left turns under a collective term *stop-points*. If $\mathscr{C}$ and $\mathscr{D}$ are two paths such that all stop-points of $\mathscr{C}$ lie on the left shore of $\mathscr{D}$ or $\mathscr{D}$ itself, we write $\mathscr{C} \precsim \mathscr{D}$. This implies $d(\mathscr{C}) \leq d(\mathscr{D})$. Let $\mathbf{S}$ be a closed rectangle in the plane bounded by lines $X = x_0$, $X = x_1$, $Y = y_0$, $Y = y_1$, $x_0 < x_1$, $y_0 < y_1$, so that all points from $\mathcal{P}$ are strictly inside $\mathbf{S}$. We write $\mathscr{A} \doteq \mathscr{B}$, if $\mathscr{A} \cap \mathbf{S} = \mathscr{B} \cap \mathbf{S}$. Naturally, the structure of the path outside $\mathbf{S}$ does not affect its relation towards $\mathcal{P}$.

Let $n \in \mathbb{N}$. Suppose that a set $\mathcal{P}$, $|\mathcal{P}| = n$, is inseparable. First, we will show that its structure can be described with a help of four finite sequences of positive integers as described in Proposition 2 below.

Consider a horizontal path $y = K$, $K \in \mathbb{Z}$. Horizontal paths (going left to right) are both uphill and downhill. If K is large enough, a path has a balance $-n$. Similarly, a horizontal line $y = -L$, $L \in \mathbb{Z}$, has a balance n, if L is large enough. Since the sequence $d\big([y = Y]\big)$ for Y ranging from K to $-L$ is non-decreasing, there exists a place where this sequence changes sign. By our assumption no path can have a balance 0. This implies the existence of $M \in \mathbb{Z}$ such that

$$d\Big([y = M + 1]\Big) < 0, \quad d\Big([y = M]\Big) > 0. \tag{1}$$

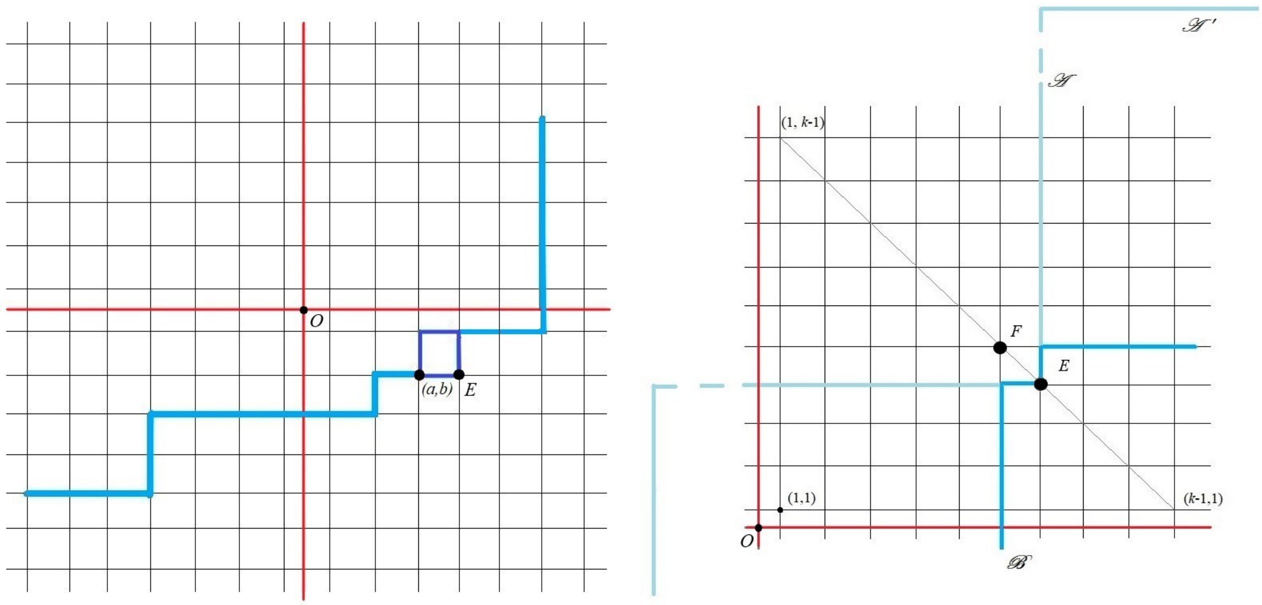

FIGURE 2. Quartering of the set $\mathcal{P}$

Let us draw a red line $y = M + \frac{1}{2}$. In the same manner, consider vertical lines going upwards. They are uphill. Analogously we can find $T \in \mathbb{Z}$, such that

$$d\Big([x = T]\Big) < 0, \quad d\Big(x = [T + 1]\Big) > 0. \tag{2}$$

Let us draw a vertical red line $x = T + \frac{1}{2}$, too. Let O be intersection of both of these. This unambiguous construction quarters points in $\mathcal{P}$. [1] The group of isometries of the plane which leave the union of two red lines intact is a dihedral group $\mathbb{D}_4$. If $\sigma \in \mathbb{D}_4$, then trivially $\sigma(\mathcal{P})$ is indivisible, too.

2. Representations of inseparable sets

We now define a procedure on the path called *a shift*, which can be *negative* or *positive*. Take any left-turn of the uphill path. Two unit segments at this stop-point are $(a, b) \to (a + 1, b)$ and $(a+1, b) \to (a+1, b+1)$. Let us replace them by $(a, b) \to (a, b+1)$ and $(a, b+1) \to (a+1, b+1)$ (see Figure 2, left, left-turn at E). After this replacement the balance of the path changes by 0, -1, or -2. This is a *negative shift*. Let us also define a negative shift for a downhill path. Let us also take a left turn. Two segments $(a, b) \to (a, b - 1)$ and $(a, b - 1) \to (a + 1, b - 1)$ are replaced by $(a, b) \to (a + 1, b)$ and $(a + 1, b) \to (a + 1, b - 1)$. This also changes the balance by 0, -1, or -2. On the other hand, if we consider right-turns, we can similarly introduce the notion of a *positive shift*. The latter changes the balance by 0, 1, or 2.

Let us return to the quartering of the set $\mathcal{P}$. Consider, for example, the first (top right) quadrant. Let the lattice points there have coordinates (x, y), $x, y \in \mathbb{N}$ (see Figure 2, right). For fixed $k \in \mathbb{N}$, $k \geq 2$, take an integer diagonal $\{(x, k - x) : 1 \leq x \leq k\}$. Call this set $\Upsilon_{1,k}$. We will now prove the following statement.

[1]This method also provides an algorithm for finding a friendly path. If vertical or horizontal line can be such, we are done. In not, there exists the unique quartering of points in $\mathcal{P}$. The second step of the algorithm will be clear from the analysis in Section 2.

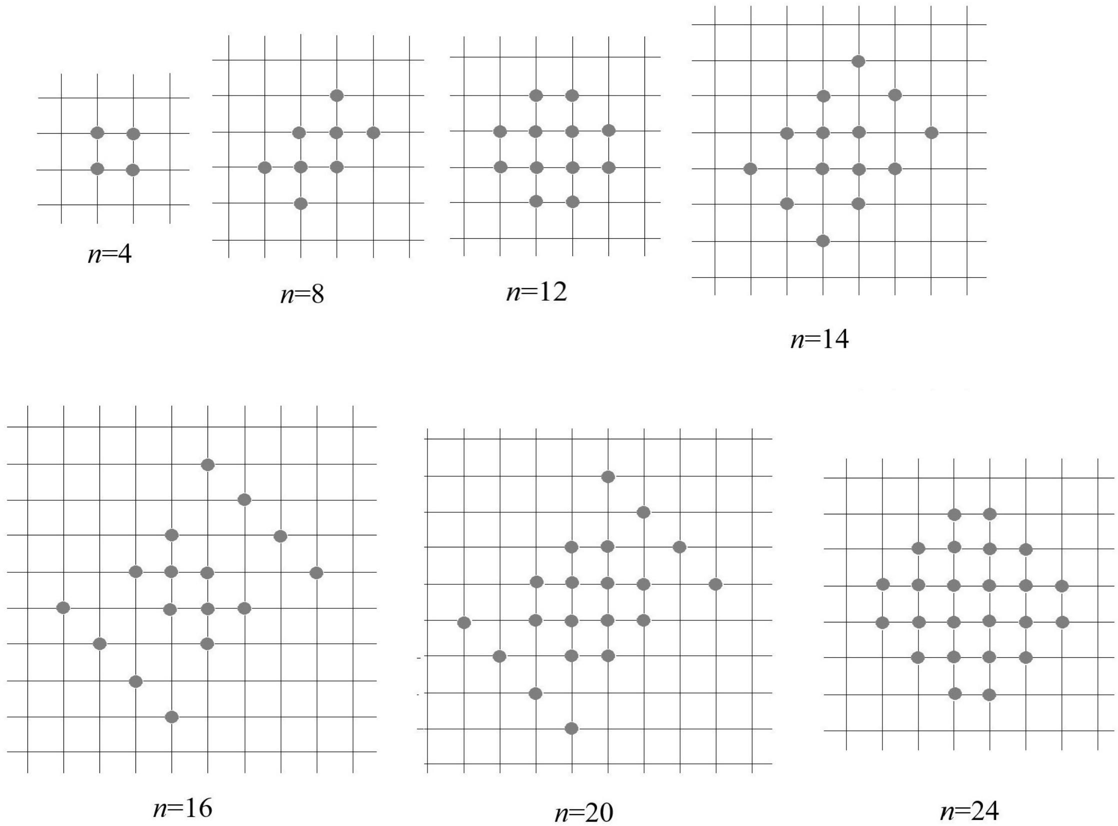

FIGURE 3. Examples

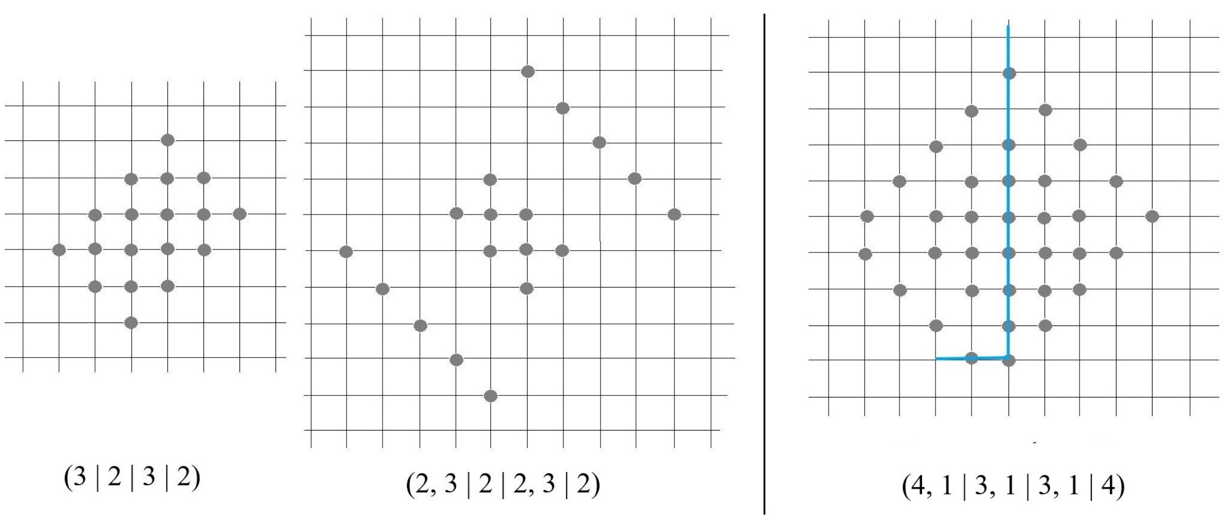

FIGURE 4. Two examples for $n = 18$ and non-example for $n = 35$. Notation comes from Proposition **??**.

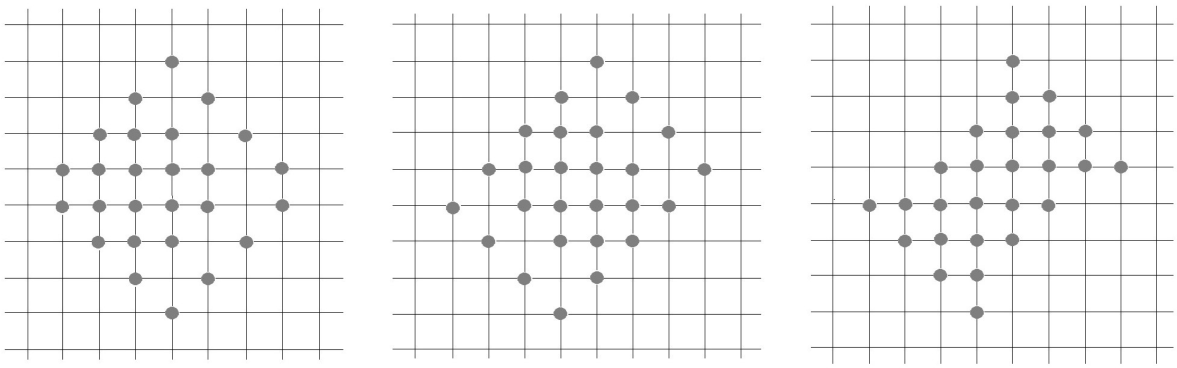

FIGURE 5. Three indivisible sets for $n = 26$

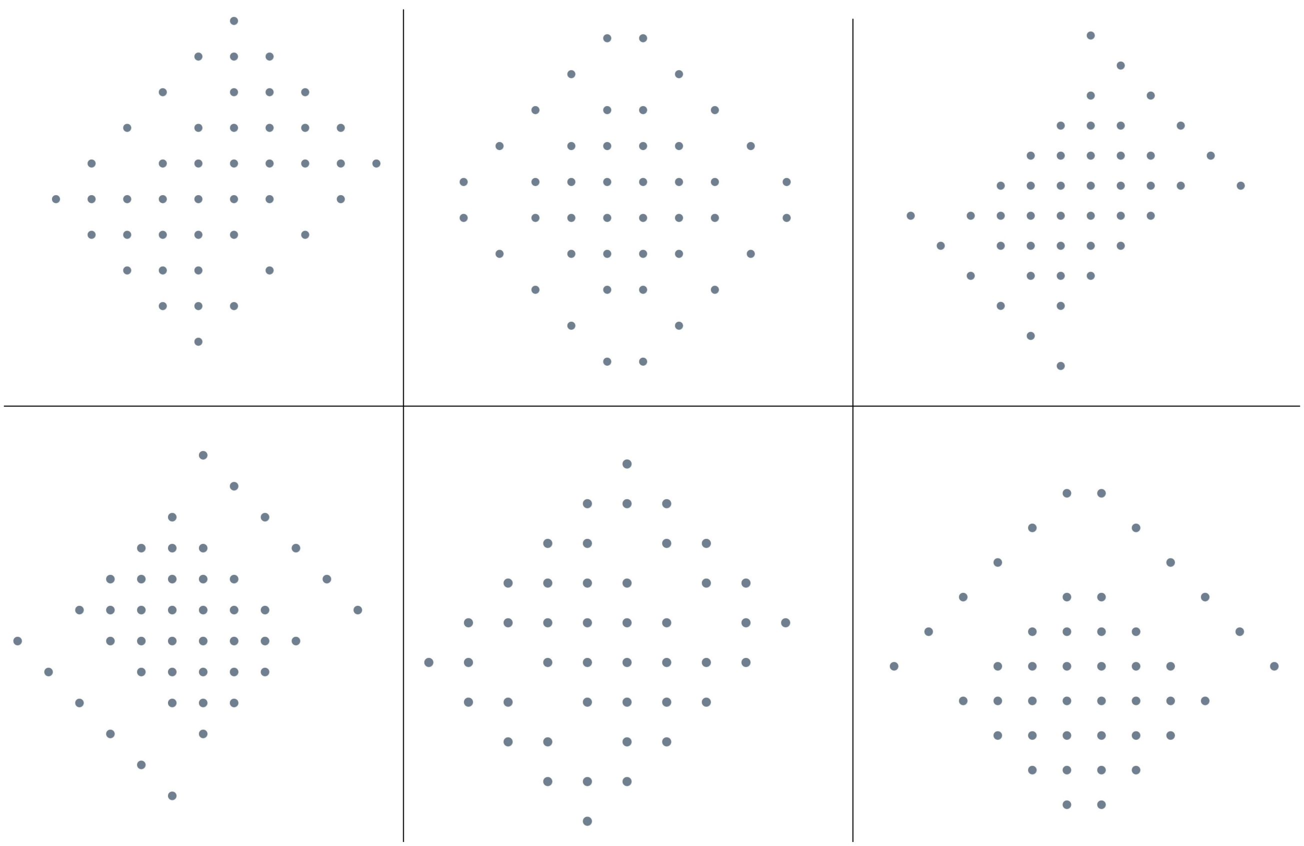

FIGURE 6. Six indivisible examples for $n = 44$

Lemma. *All or none integer points of this diagonal belong to $\mathcal{P}$.*

Suppose the opposite: there exists such k and two points $E = (x, y)$, $F = (x - 1, y + 1)$, such that $E \in \mathcal{P}$, $F \notin \mathcal{P}$. From the symmetry considerations (minding the remark concerning the group $\mathbb{D}_4$) we may assume $x \geq y$.

Consider now two uphill paths $\mathscr{A}$ and $\mathscr{B}$, as marked in the Figure 2, right. Here both dotted half-lines extend to infinity. Outside $\mathbf{S}$, the path $\mathscr{A}$ can be altered to attain a shape of a path

$\mathscr{A}'$. This does not change the balance. Now, the path $\mathscr{A}'$ can be gradually transformed into the the path $\mathscr{B}$ by a series of positive shifts, leaving the left-turn at E intact:

$$\mathscr{A}' = \mathscr{E}_0 \precsim \mathscr{E}_1 \precsim \cdots \mathscr{E}_r \doteq \mathscr{B}.$$

Moreover, since $\mathscr{A} \precsim [y = M + 1]$, we have $d(\mathscr{A}') = d(\mathscr{A}) < 0$. In a similar vein, since $[x = T + 1] \precsim \mathscr{B}$, $d(\mathscr{B}) > 0$. By our assumption, a sequence $\{d(\mathscr{E}_i) : 0 \leq i \leq r\}$ is monotone, does not contain 0 and makes a jump at most by 2. Since it goes from negative to a positive value, there exists s, $1 \leq s \leq r$, such that $d(\mathscr{E}_{s-1}) = -1$ and $d(\mathscr{E}_s) = 1$. Consider the path $\mathscr{E}_s$. Making a negative shift for $\mathscr{E}_s$ at E produces a path with a balance 0. A contradiction. ■

The kth diagonal $\Upsilon_{r,k}$ of a quadrant r, $1 \leq r \leq 4$, is given by $\tau^{r-1}(\Upsilon_{1,k})$, where $\tau \in \mathbb{D}_4$ is a rotation by $\frac{\pi}{2}$ rotation around O. Trivially, the property of $\Upsilon_{1,k}$ we have just demonstrated (that is, all or none of its points belong to $\mathcal{P}$) holds for any $\Upsilon_{r,k}$, too. Therefore, we have proved the following.

Proposition 2. *Every indivisible set $\mathcal{P}$ can be represented in the form*

$$\begin{aligned} \theta = [L_1 L_2 L_3 L_4], \qquad & L_r = (v_{r,1}, v_{r,2}, \ldots, v_{r,k_r}), \\ 1 \leq v_{r,1} < v_{r,2} < \cdots < v_{r,k_r}, \qquad & r \in \{1,2,3,4\},\ k_r \in \mathbb{N}. \end{aligned} \tag{3}$$

Here $1,2,3$ and 4 correspond, respectively, to the namesake quadrant. With this notation, integer diagonals with numbers $v_{r,i}$ fully belong to $\mathcal{P}$, and these exhaust all points. In this notation,

$$n = \sum_{r=1}^{4} \sum_{t=1}^{k_r} v_{r,t}.$$

Naturally, only specific choices of such representation of n produce indivisible configurations (this will be treated in the next Section). Few examples. The smallest integer with two indivisible configurations is $n = 18$ (see Figure 4), where these collections are, respectively,

$$\theta_1 = [(1,2,3)(1,2)(1,2,3)(1,2)],\ \theta_2 = [(1,5)(1,2)(1,5)(1,2)].$$

The smallest integer with three indivisible configurations is $n = 26$ (Figure 5), where these collections are

$$\begin{aligned} \theta_1 &= [(1,2,4)(1,2,3)(1,2,4)(1,2,3)], \\ \theta_2 &= [(1,2,4)(1,2,3)(1,2,3)(1,2,4)], \\ \theta_3 &= [(1,2,3,4)(1,2)(1,2,3,4)(1,2)]. \end{aligned}$$

This finishes theoretical part of the paper. All we have to do is to write conditions for inseparability, and produce a corresponding computer code. We do it with MAPLE in the next section.

3. Full conditions for indivisibility and computer code

So, our task now is as follows. Assume, we run a computer program which enumerates all possible quadruples $[L_1 L_2 L_3 L_4]$ of finite increasing integer sequences. Which of them truly represent an indivisible set? Thre tests are needed to ensure that.

`Test A.`

First, we must check that the quartering of the set is proper. That is,

1. $-N_3 - N_4 < Y_1 + Y_2 - Y_3 - Y_4 < N_1 + N_2$,

2. $-N_2 - N_3 < Y_1 + Y_4 - Y_2 - Y_3 < N_1 + N_4$.

Of course, in case of necessity using reflection with respect to both red lines, we can always achieve $Y_1 + Y_2 \geq Y_3 + Y_4$ and $Y_1 + Y_4 \geq Y_2 + Y_3$. Yet, for computational purposes, it is more convenient to work with conditions **1.** and **2.** directly, and to collect all symmetric configurations as representatives of the one afterwards.

3.1. **Example with** $n = 108$**.** To better illustrate the way we check that whether the given configuration θ represents an inseparable set, consider the following example. Let $\theta = [L_1 L_2 L_3 L_4]$, where

$$L_1 = (1,2,3,5,6,9), \quad L_2 = (1,2,3,6,7,9), \quad L_3 := (1,2,3,4,6,9), \quad L_4 := (1,2,4,6,7,9).$$

Here $Y = (26, 28, 25, 29)$, $N = (6, 6, 6, 6)$ (see Figure 7, left).

Case 3. Assume, there exists a friendly line which enters the configuration at $\mathcal{Q}_2$, exists at $\mathcal{Q}_4$, and also passes through $\mathcal{Q}_1$. Thus, friendly line does not pass through $\mathcal{Q}_3$, and this is exactly what the name "**Case 3**" means.

(-) Consider a down-hill path which takes a single right-turn at the point $(1;1,1)$ (Figure 7, top-left). There are $N_2 + N_4 + 1 = 13$ points on this path. Thus, it cannot be friendly.

(3-i) We now ask: which down-hill paths making a single right-turn at $(1;x_0,1)$ contain an even number of points (only these potentially can be friendly). In this particular example, the answer is $x_0 = 6$ (Figure 7, top-right). In this case $T(\ell) = 14$, $R(\ell) = 47$, $L(\ell) = 47$ (by a lucky chance, this is a friendly line).

(3-ii) In the same vein, if a down-hill path makes a single right-turn at $(1;,1,y_0)$ and contains an even amount of points, the smallest such u_0 is $y_0 = 4$ (same picture, bottom-left). In this case $T(\ell) = 12$, $R(\ell) = 40$, $L(\ell) = 56$.

(3-iii) As a final sub-case, any down-hill path making a single right-turn at points $(1;3,2)$ or $(1;2,3)$ contains 12 points. In both cases, $R(\ell) = 44$, $L(\ell) = 52$.

Now, positive shifts made strictly inside $\mathcal{Q}_2$ or $\mathcal{Q}_4$ alter a balance by 0 or 2 (1 is excluded do to construction). Thus, in all cases where $R(\ell) < L(\ell)$, we can construct a friendly path by a series of positive shifts. In case **(ii)** and **(iii)**, these (from the deviation point onwards) are shown in sky-blue.

This gives a clear picture how one checks whether the given set is indivisible: in **Cases 1, 3** and all sub-cases (where we have a down-hill path), inequality $R(\ell) > L(\ell)$ must hold. Analogously, $L(\ell) > R(\ell)$ must hold in **Cases 2, 4**. Analogous inequalities should hold in all **Cases 1, 2** and **4**.

3.2. **General check.** Thus, the following collection of numbers will play an exceptional in calculations. Consider even n and **Case 3-i**. In the example presented in Subsection 3.1 , this number exactly corresponds to x_0; that is, 6.

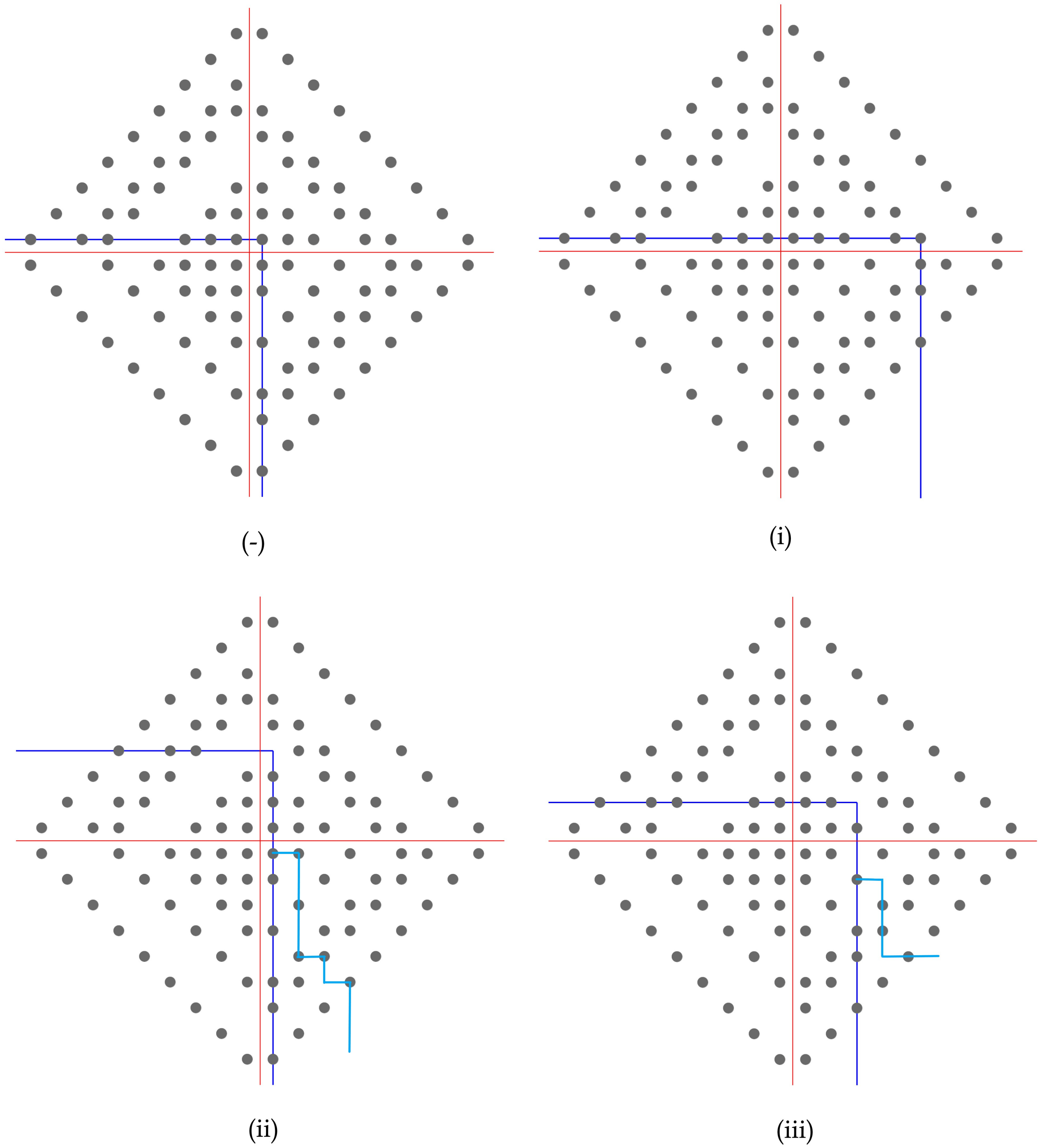

FIGURE 7. An example with $n = 108$

This leads to a general definition Analogously, $w_{1,1}$ be the smallest $p \in \mathbb{N}$, not contained in L_1. This number will play a crucial role in counting the number of configurations.

Definition 1. *For each ordered pair $(\mathcal{Q}_i, \mathcal{Q}_j)$ of neighbouring quadrants $(i - j \equiv \pm 1 \pmod 4)$, Let $w_{i,j}$ be the smallest $p \in \mathbb{N} \setminus \{1\}$ such that*

a) *either* $p \in L_i$, $p-1 \notin L_j$,
b) *or* $p \in L_i$, $p-1 \in L_j$.

Looking back at Subsection 3.1, we see that $w_{1,4} = 6$, and this corresponds exactly sub-case **3-ii** and the point $x_0 = 6$. MAPLE program which checks this criteria is as follows. The output is $[\mathtt{Ls}, \mathtt{Rs}, \mathtt{yra}, \mathtt{Tt}]$, and $\mathtt{Ls} > \mathtt{Rs}$ must be fulfilled for all pairs $(i - j \equiv \pm 1 \pmod 4)$.

Test B.

```
B:=proc(j,i::integer)
local Tt, t, wij, g, h, lik, v, Ls, Rs, yra;
global L,N,n,m;
yra:=true;
t:=2;
while member(t,L[i])=member(t-1,L[j]) and t<=max(m[i],m[j]) do t:=t+1: end do:
if member(t,L[i]) then Tt:=N[modp(j+1,4)+1]+N[j]+2: end if:
if member(t-1,L[j])then Tt:=N[modp(j+1,4)+1]+N[j]: end if:
if t=max(m[i],m[j])+1 then yra:=false: end if:
if yra then
wij:=t:
if L[i][N[i]]<=wij then h:=N[i] else
h:=1:
while (L[i][h]<=wij) do h:=h+1: end do:
h:=h-1:
end if:
g:=Tt-h-N[modp(j+1,4)+1]:
lik:=0:
for v from 1 to N[j] do if L[j][v]>wij then lik:=lik+(L[j][v]-wij): end if: end do:
Ls:=Y[modp(i+1,4)+1]+Y[j]-g-lik:
Rs:=n-Tt-Ls:
else
Ls:=1: Rs:=0: end if:
[Ls,Rs,yra,Tt]:
end proc:
```

Let us now count. In sub-sub-case **a**), $T(\ell) = N_i + N_j + 2$, in sub-sub-case **b**), $T(\ell) = N_i + N_j$.

Definition 2. *For quadrant* $\mathcal{Q}_i$*, let* r_i *be the smallest* $p \in \mathbb{N}$ *not contained in* L_i*.*

Test C.

```
C:=proc(i::integer)
local s, t,lik, Tt, v, Rs, Ls, Ger;
global L,N,n,m;
s:=1;
while member(s,L[i]) do s:=s+1: end do:
Ger:=true:
for t from 1 to s-2 do
lik:=0:
```

```
Tt:=0:
if member(s-t,L[modp(i,4)+1]) then Tt:=Tt+1: end if:
for v from 1 to N[modp(i,4)+1] do if L[modp(i,4)+1][v]>s-t then
lik:=lik+(L[modp(i,4)+1][v]-s+t): Tt:=Tt+1: end if: end do:
if member(t+1,L[modp(i-2,4)+1]) then Tt:=Tt+1: end if:
for v from 1 to N[modp(i-2,4)+1] do if L[modp(i-2,4)+1][v]>t+1 then
lik:=lik+(L[modp(i-2,4)+1][v]-t-1): Tt:=Tt+1: end if: end do:
#end do:
Rs:=lik+Y[i]-(t+1)*(s-t)+1:
Tt:=Tt+s-1:
Ls:=n-Tt-Rs:
Ger:=Ger and (Ls>Rs):
end do:
Ger
end proc:
```

As an example, Figure 6 shows six possible indivisible configurations of $n = 44$ points.

We finish with the following statement which slightly improves the part **a** of the initial problem in Monthly.

Proposition 3. *All even numbers for which there exists indivisible set $\mathcal{P}$ with $|\mathcal{P}| = n$ of the form $(a\,|\,b\,|\,c\,|\,d)$ are given by*

$$n = a^2 + a + b^2 + b,\ a, b \in \mathbb{N},\ b \leq a \leq b + \sqrt{2b} + \frac{1}{2}.$$

Such representation of n, if it exists, is unique. The sequence of these numbers starts from

$$4, 8, 12, 18, 24, 26, 32, 40, 42, 50, 60, 62, 72, 76, 84, 86, 98, 102, 112, 114, 128, 132, 144, 146, 162,$$
$$166, 180, 182, 188, 200, 204, 220, 222, 228, 242, 246, 264, 266, 272, 288, 292, 312, 314, 320, 338, 342.$$

We always have in mind that for integers $a \geq b \geq 1$, the condition in Proposition is just a re-write of the implicit condition $(a-b)^2 < a+b$. The smallest difference between these terms is 2, and it corresponds to the identity

$$(a^2 + a + a^2 + a) + 2 = (a+1)^2 + (a+1) + (a-1)^2 + (a-1).$$

Proof. The uniqueness part follows from the following result.

Lemma. *Given a positive integer n. Subject to an additional restriction $(a-b)^2 < 4(a+b)+4$, there exists at most two representations of n in the form $n = a^2 + b^2$, $a, b \in \mathbb{N}$.*

Proof. Let $n = a^2 + b^2$, $a - b = Q$, $a + b = P$. Then

$$a = \frac{P+Q}{2},\ b = \frac{P-Q}{2}, \quad P^2 + Q^2 = 2n^2.$$

Note that both numbers P, Q are of the same parity. In terms of P, Q, the condition on (a, b) rewrites as $Q^2 < 4(P+1)$. Next, note that if a pair (a, b) provides a representation of n in the form $n = a^2 + b^2$, so does (b, a).

Suppose, there exists three such representations. Consequently, at least two of them additionally satisfy $a \geq b$. This implies $Q \geq 0$. Let (a_1, b_1), (a_2, b_2) be these two pairs, corresponding to

(P_1, Q_1), and P_2, Q_2. Assume $P_1 > P_2$, $0 \leq Q_1 < Q_2$. First, we have

$$P_1^2 + Q_2^2 = P_2^2 + Q_2^2.$$

This identity modulo 4 shows that all numbers present are of the same parity. With this remark in mind, note that

$$Q_2^2 \geq Q_2^2 - Q_1^2 = (P_1 - P_2)(P_1 + P_2) \geq 2(P_1 + P_2) \geq 2(2P_2 + 2),$$

an this is a contradiction. An identity

$$(2n^2 + 2n + 1)^2 + (2n^2 + 2n + 1)^2 = (2n^2 - 1)^2 + (2n^2 + 4n + 1)^2, \quad n \in \mathbb{N}$$

shows that the inequality in Lemma is sharp. □

Now, assume $n = a^2 + a + b^2 + b$, $b \leq a$. If we rewrite this representation as $4n + 2 = (2a + 1)^2 + (2b + 1)^2$. Lemma shows that that such representation, subject to the condition $(a - b)^2 < 2a + 2b + 3$, is unique. This covers the narrower case $(a - b)^2 < a + b$. □

References

[1] Problems and solutions, *Amer. Math. Monthly* **117** (2) February (2010), 182.

[2] Giedrius Alkauskas Friendly paths, *Amer. Math. Monthly* **119** (2) February (2012), 167–168.

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