

NARROWEST INTERVALS CONTAINING TWO PAIRS OF INTEGERS WITH EQUAL CUBE SUMS

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ABSTRACT. In this note, we define several new sequences related to taxi-cab numbers and cubic Euler's equation.

1. CALCULATIONS FOR EULER'S EQUATION

Euler's cubic equation is an equation $x^3 + y^3 = u^3 + v^3 = N$ to be solved in integers. For example, $4104 = 2^3 + 16^3 = 9^3 + 15^3$ is the solution.

Definition 1 (Sequence A_1). For any positive integer x , let $y \in \mathbb{N}$, $y > x$, be the smallest positive integer such that there exists $u, v \in \mathbb{N}$, $x < u < v < y$, so that

$$x^3 + y^3 = u^3 + v^3.$$

We define $A_1(x) = y$.

Since the identity $x^3 + (12x)^3 = (9x)^3 + (10x)^3$ holds, it follows that $y \leq 12x$. This sequence is not yet contained yet in OEIS. Here is the first 26 terms¹ (the offset is 1):

12, 16, 36, 32, 60, 48, 84, 53, 34, 27, 93, 40, 156, 112, 80, 106, 39, 68, 228, 54, 238, 176, 94, 80, 67, 156.

Let $R(x) = \frac{y}{x}$. As we have seen, $R(x) \leq 12$. Figure 1 shows the pointplot of this function for integral x , $1 \leq x \leq 1200$.

Problem. We ask whether there exist the limit

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{x=1}^n R(x).$$

This problem will be addressed in next version of this note.

Definition 2 (Primitive quadruples). Let us consider only those $x \in \mathbb{N}$ for which the corresponding quadruple $(x, y; u, v)$ satisfies an additional condition $\text{GCD}(x, y, u, v) = 1$. The ordered sequence of these x 's is the sequence A_2 .

Here are the first 30 terms:²

1, 2, 8, 9, 10, 11, 12, 15, 17, 21, 23, 25, 26, 29, 30, 31, 32, 35, 37, 41, 42, 47, 49, 50, 51, 53, 55, 57, 61, 62.

Definition 3 (GCD of a quadruple). Let $A_3(x) = \text{GCD}(x, y, u, v)$.

$A_3(n)$ is thus a divisor of n . Here is the beginning of this sequence:³

1, 1, 3, 2, 5, 3, 7, 1, 1, 1, 1, 1, 13, 7, 1, 2, 1, 2, 19, 2, 1, 11, 1, 2, 1, 1, 3, 14,

In this notation, sequence A_2 contains all n (in the increasing order) satisfying $A_3(n) = 1$.

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¹1200 terms are contained in this file: <https://web.vu.lt/mif/g.alkauskas/math/cubic-sums.txt>

²All 592 terms ≤ 1200 are here: <https://web.vu.lt/mif/g.alkauskas/math/cubic-sums2.txt>

³1200 first terms are here: <https://web.vu.lt/mif/g.alkauskas/math/cubic-sums3.txt>

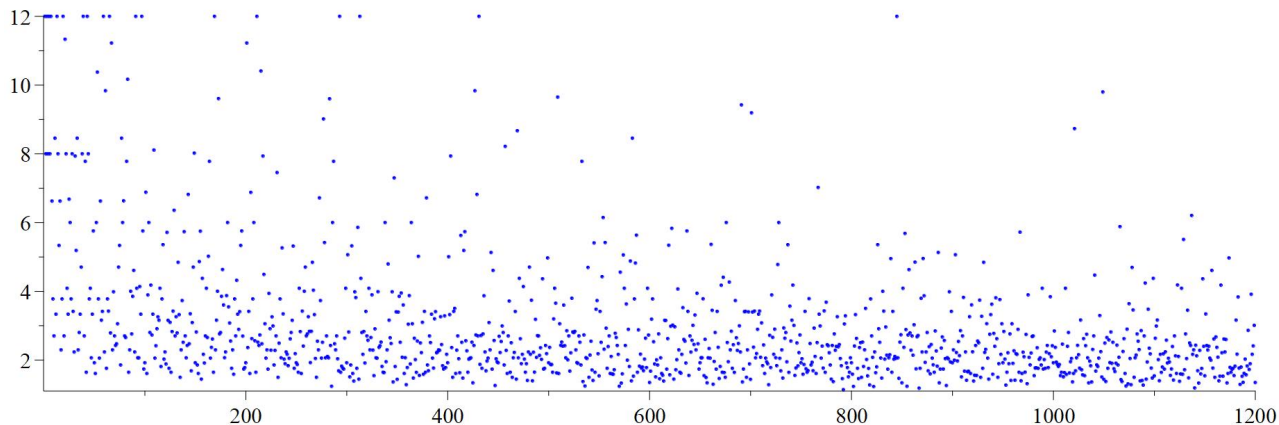


FIGURE 1. Pointplot for the function $R(x)$, $1 \leq x \leq 1200$.

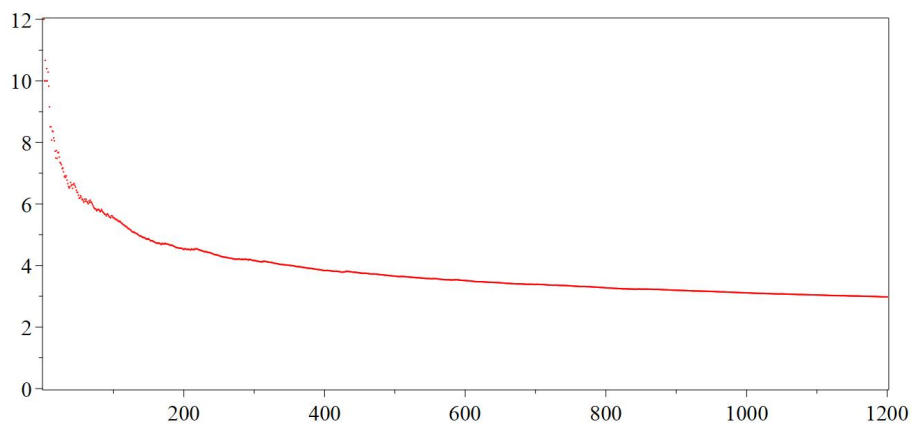


FIGURE 2. Pointplot for the function $(\sum_{n=1}^x R(n))/x$, $1 \leq x \leq 1200$.

Definition 4 (Sequence of minima records). *Let us define the sequence $A_4(n)$ as the n -th minima record of the function $R(x)$.*

Obviously, this is a sub-sequence of A_2 .

Now we will define yet another object.

Definition 5. *For each $x \in A_2(\mathbb{N})$ (this is the set defined by the sequence A_2), define a set $\mathfrak{S}(x)$ by the condition $N \in \mathfrak{S}(x)$, if*

$$A_3(Nx) = N.$$

Consider for example $x = 1$. The condition requires $A_3(N) = N$. That is, in the quadruple $(N, y; u, v)$, all terms are divisible by N . Consequently, the quadruple is $(N, 12N, 9N, 10N)$. For N in the range $[1, 1200]$, the following terms belong to $\mathfrak{S}(1)$:

$$1, 3, 5, 7, 13, 19, 39, 43, 59, 65, 91, 97, 169, 211, 293, 313, 431, 845.$$

We thus have a computational result.

Proposition 1. *Let $x = 845$. Then the smallest integral $y > x$ such that there exists integers $x < u < v < y$ with $x^3 + y^3 = u^3 + v^3$ is equal to $12 \cdot 845$.*

Sets \mathfrak{S}		
n	$A_2(n) = x$	$\mathfrak{S}(x)$
1	1	$1\{1, 3, 5, 7, 13, 19, 39, 43, 59, 65, 91, 97, 169, 211, 293, 313, 431, 845\}$
2	2	$\{1, 2, 3, 7, 11, 14, 19, 22\}$
3	8	$\{1, 2, 7\}$
4	9	$\{1, 2, 3, 5, 6, 7, 13, 21, 23, 26, 65, 67, 91\}$
5	10	$\{1, 2, 4, 7, 8, 13, 14, 16, 19, 23, 26, 28, 29, 32, 41, 43, 46, 49, 52, 53, 56, 64, 82, 86, 91, 98, 104, 109, 112\}$
6	11	$\{1, 3, 7, 53\}$
7	12	$\{1, 2, 3, 4, 8, 9, 16, 32, 59, 64\}$
8	15	$\{1, 5, 13\}$
9	17	$\{1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20, 20, 23, 26, 31, 37, 38, 40, 41, 43, 53, 67\}$
10	21	$\{1\}$
11	23	$\{1, 2, 4, 5, 7, 11, 13, 14, 22, 25, 26, 37, 43, 44, 49\}$
12	25	$\{1\}$
13	26	$\{1, 2, 3, 4, 7, 8, 11, 13, 14, 26, 28\}$
14	29	$\{1, 2, 7, 11, 13, 14, 19, 22, 23, 26, 29, 31\}$
15	30	$\{1, 2, 3, 4, 6, 8, 13, 16, 23, 29, 39\}$
16	31	$\{1, 7, 13\}$
17	32	$\{1, 13\}$

TABLE 1. Computations

Problem. *Is the set $\mathfrak{S}(1)$ infinite?*

2. EXPLICIT FORMULAS

Complete list of integer solutions to $x^3 + y^3 = u^3 + v^3$ was given by Choudhry [1], and they read as

$$\begin{cases} x = c(b - a)(a^2 + ab + b^2) + c^4, \\ y = (a^2 + ab + b^2)^2 + (2a + b)c^3, \\ u = (a^2 + ab + b^2)^2 + (b - a)c^3 \\ v = (a^2 + ab + b^2)(2a + b)c + c^4, \end{cases}$$

with additional allowance to factor out GCD of the quadruple x, y, u, v . Consider a carefully chosen special example $a = N - 1$, $b = N^2 - N + 2$, $c = N^2 - N + 1$. After factoring out c^2 , we obtain

$$\begin{cases} x = 2N^4 - 4N^3 + 9N^2 - 8N + 10, \\ y = 2N^4 + 6N^2 + N + 9, \\ u = 2N^4 - 3N^3 + 12N^2 - 5N + 12, \\ v = 2N^4 - N^3 + 6N^2 + N + 1. \end{cases}$$

Since $\frac{y}{x} \approx 1 + \frac{2^{5/4}}{x^{1/4}}$, this shows that $\liminf_{x \rightarrow \infty} R(x) = 1$.

Corollary. *The sequence A_4 is infinite.*

Here is MAPLE code which does all computations up to $x \leq 1200$. The array $\text{Es1}[W]$ (W is a permutation of $\{1, \dots, 1200\}$) is the sequence x 's: 1, 2, 3, The array $\text{Es2}[W]$ is the sequence of y 's: 12, 16, 36, 32, 60, 48, ... (that is, A_1). $\text{Es3}[W]$ is u 's, $\text{Es4}[W]$ is v 's, and $\text{Es5}[W]$ is the sequence A_3 .

Sequence A_4		Supplementary sequences				
n	$X(n)$	$Y(n)$	$Z(n)$	$W(n)$	$a^3 + b^3 = c^3 + d^3$	$R(X)$
1	1	12	9	10	1729	12
2	2	16	9	15	4104	8
3	8	53	29	50	149389	6.625
4	9	34	16	33	40033	3.7777
5	10	27	19	24	20683	2.7
6	17	39	26	36	64232	2.2941
7	30	67	51	58	327763	2.2333
8	42	69	56	61	402597	1.6428
9	51	82	64	75	684019	1.6078
10	135	202	166	183	10702783	1.4962
11	156	225	169	218	15187041	1.4423
12	285	352	309	334	66763333	1.2350
13	792	901	829	870	1228225789	1.1376
14	1634	1858	1707	1797	10776828816	1.1370
15	3751	4172	3878	4063	125392670199	1.1122
16	4026	4267	4104	4195	142946631739	1.0598
17	6192	6555	6259	6494	519062033763	1.0586
18	14934	15697	15090	15553	7198320746377	1.0510
19	15768	16533	16130	16189	8439526756269	1.0485
20	16147	16626	16299	16480	8805759770899	1.0296
21	45121	46440	45238	46329	192017975290561	1.0292
22	58230	59467	58683	59026	407736961367563	1.0212
23	61389	62274	61733	61936	472852910313693	1.0144
24	79876	80937	80209	80610	1039824823550329	1.0132

TABLE 2. Computations

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restart:
N:=0:
Por:=90660270:
Sk:=1200: #upper bound for a(n)
f:=Array(1..Por, datatype=integer):
Ar:=Array(1..Por, datatype=integer):
Br:=Array(1..Por, datatype=integer):
fc1:=Array(1..Por,datatype=integer[2]):
fc2:=Array(1..Por,datatype=integer[2]):
Es1:=Array(1..Sk, datatype=integer[2]):
Es2:=Array(1..Sk, datatype=integer[2]):
Es3:=Array(1..Sk, datatype=integer[2]):
Es4:=Array(1..Sk, datatype=integer[2]):
Es5:=Array(1..Sk, datatype=integer[2]):
Rado:=Array(1..Sk,datatype=boolean):
for t from 1 to Sk do Rado[t]:=false end do:
for X from 1 to Sk do
for Y from X to 12*X do

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N:=N+1: f[N]:=Y^3+X^3: fc1[N]:=Y: fc2[N]:=X:
end do:
end do:
for X from Sk+1 to floor(9.53*Sk) do
for Y from X to 12*Sk do
N:=N+1: f[N]:=Y^3+X^3: fc1[N]:=Y: fc2[N]:=X:
end do:
end do:
Ar,Br:=sort(f,'output=[sorted,permutation]'):
K:=0:
for i from 1 to N-1 do if (Ar[i]=Ar[i+1]) and (fc2[Br[i]]<=Sk) then
if (not Rado[fc2[Br[i]]]) then
Rado[fc2[Br[i]]]:=true:
K:=K+1:
Es1[K]:=fc2[Br[i]]:
Es2[K]:=fc1[Br[i]]:
Es3[K]:=fc2[Br[i+1]]:
Es4[K]:=fc1[Br[i+1]]:
Es5[K]:=igcd(fc1[Br[i]], fc2[Br[i]], fc1[Br[i+1]], fc2[Br[i+1]]):
end if:
end if:
end do:
Ess:=Array(1..K):
for k from 1 to K do Ess[k]:=Es1[k] end do:
W:=sort(Ess,'output=permutation'):

```

REFERENCES

1. CHOUDHRY, AJAI, On equal sums of cubes, *Rocky Mountains Journal of Mathematics*, **28** (4), 1998, <https://www.jstor.org/stable/44238229>.
2. ONO, KEN, TREBAT-LEDER, SARAH, The 1729 K3 surface, *Research in Number Theory*, **2** (26), 2016, <https://link.springer.com/article/10.1007/s40993-016-0058-2>.

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