COVERING SYSTEMS OF $[1, M-1]$ WHICH AVOID M AND $M+1$

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Denote arithmetic progression (AP) $\{c + nd : n \in \mathbb{Z}\}$ by (c, d). Define $b(k)$ as the largest positive integer M for which there exist k APs which cover all integers in the range $[1, M-1]$, but omit M. According to the theorem Crittenden and Vanden Eynden [1], $b(k) = 2^k$; the example is given by k progressions $A_i = (2^{i-1}, 2^i)$, $1 \leq i \leq k$. A very transparent proof (as if from "The Book") was given by Balister, Bollobás, Morris, Sahasrabudhe and Tiba [2].

In such formulation, the next logical step is to define the following sequence.

Definition. *Let $a(k)$ be the largest positive integer M for which there exist k APs which cover all integers in the range $[1, M-1]$, but omit M and $M+1$.*

Table 1 list first 15 initial values. The calculation of the last one lasted for several hours. The following is one of the example of extremal collection of APs for $k = 15$:

$$(1,3); (2,4); (2,5); (3,5); (1,5); (4,7); (4,8); (5,9);$$
$$(2,9); (5,10); (9,15); (8,16); (12,19); (8,27); (26,27).$$

So, with this approach few more values can be calculated.

Sequence $a(n)$															
k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$a(k)$	**2**	**3**	**5**	**11**	**17**	**23**	**36**	**54**	**89**	**119**	**179**	**239**	**360**	**495**	**719**
Lower bound				11	15	23	31	47	63	95	127	191	255	383	511

Table 1. Initial terms of the sequence $a(k)$

Lower bound. It trivially follows from the result on the sequence $b(k)$ that for $k \geq 2$, $a(k) < 2^k$. Consider $k \geq 3$ arithmetic progressions A_i, $2 \leq i \leq k+1$. They cover all even integers in the range $[2, 2^{k+1}-2]$, but miss 2^{k+1}. On the other hand, arithmetic progression $2A_j - 1$, $1 \leq j \leq k$ cover all odd integers in the range $[1, 2^{k+1}-3]$, but miss $2^{k+1}-1$. Thus, there exist $2k$ APs which cover all integers in the range $[1, 2^{k+1}-2]$, but miss $2^{k+1}-1$ and 2^{k+1}. Therefore, $a(2k) \geq 2^{k+1} - 1$. However, the first collection contains a singleton and a 2-term progression; the same holds for the second collection. These 6 numbers are

$$\{2^{k-1}-1, 2^{k-1}, 2 \cdot 2^{k-1} - 1, 2 \cdot 2^{k-1}, 3 \cdot 2^{k-1} - 1, 3 \cdot 2^{k-1}\}.$$

To cover these, 4 APs are not needed, 3 will suffice:

$$\{\underline{2^{k-1}-1}, \underline{2 \cdot 2^{k-1}}, 3 \cdot 2^{k-1}+1, 2^{k+1}+2\},$$
$$\{\underline{2^{k-1}}, \underline{3 \cdot 2^{k-1} - 1}, 5 \cdot 2^{k-1} - 2\},$$
$$\{2^{k-1}-2, \underline{2^k - 1}, \underline{3 \cdot 2^{k-1}}, 2^{k+1}+1\}.$$

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Note that for $k \geq 3$, $5 \cdot 2^{k-1} - 3 \geq 2^{k+1} + 1$. So, $a(2k-1) \geq 2^{k+1} - 1$.

As a slightly different construction, consider a collection $3A_i - 2$, $1 \leq i \leq k$. These cover all integers $3n-2$ up to $3 \cdot 2^k - 5$. A collection $3A_i - 1$, $1 \leq i \leq k$ cover all integers $3n-1$ up to $3 \cdot 2^k - 4$. Adding a single AP $(0, 3)$, we cover all non-negative integers up to $3 \cdot 2^k - 3$, and miss $3 \cdot 2^k - 2$ and $3 \cdot 2^k - 1$. Thus, $a(2k+1) \geq 3 \cdot 2^k - 1$ (recall that 0 is also covered, so we shift by 1 upwards). As before, the 6 terms covered by the two collections need not 4 APs, but 3. This gives

$$a(2k) \geq 3 \cdot 2^k - 1.$$

0.1. **Asymptotics.** The initial values for the sequence $a(n)$ strongly support the fact that

$$\lim_{k \to \infty} \frac{a(k+2)}{a(k)} = 2.$$

Therefore we pose the following

Conjecture. $a(k) = \Theta\big(2^{k/2}\big)$.

References

1. R.B. Crittenden, C.L. Vanden Eynden; Any n arithmetic progressions covering the first 2^n integers cover all integers. *Proc. Amer. Math. Soc.* 24 (1970), 475–481.
2. P. Balister, B. Bollobàs, R. Morris, J. Sahasrabudhe, M. Tiba; Covering intervals with arithmetic progressions. *Acta Math. Hungar.* **161**(1) (2020), 197–200. https://link.springer.com/article/10.1007/s10474-019-00980-z.

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