CONSECUTIVE INTEGERS IN SETS $S_n = \{d + \frac{n}{d} : d|n\}$

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In this note, we solve the following new olympiad-level problem. On the way, two new integer sequences are being discovered.

Problem. *Let $n \in \mathbb{N}$. Consider the set S_n, which consists of all numbers $d + \frac{n}{d}$, where d is the divisor of n.*

Prove that there exists infinitely many n such that the set S_n contains 3 consecutive integers.

The smallest such number is $n = 144$:

$$\frac{144}{12} + 12 = 24, \quad \frac{144}{9} + 9 = 25, \quad \frac{144}{8} + 8 = 26.$$

Here are all such numbers n in the range $[1, 3 \cdot 10^5]$:

$$\mathbf{144}, \mathbf{180}, \mathbf{1260}, \mathbf{1440}, \underline{2520}, \mathbf{5040}, \mathbf{5544}, 7200, \mathbf{14040}, \underline{\mathbf{15120}}, 25200, \mathbf{31680}, \mathbf{33660},$$
$$37800, 46800, 59400, \mathbf{62244}, \mathbf{65520}, \underline{70560}, 83160, 107100, \mathbf{110880}, \mathbf{115920}, 166320, \quad (1)$$
$$\underline{169344}, 176400, 180180, \mathbf{183600}, \mathbf{190944}, 221760, 277200, \mathbf{287280}, \mathbf{297540}.$$

(Ignore bold and underlined values; these marks are for the sake of **Solution 1**). It can be proved that all numbers above are divisible by 36. Performing the division, another sequence is obtained:

$$4, 5, 35, 40, 70, 140, 154, 200, 390, 420, 700, 880, 935, 1050, 1300, 1650, 1729, 1820,$$
$$1960, 2310, 2975, 3080, 3220, 4620, 4704, 4900, 5005, 5100, 5304, 6160, 7700, 7980, 8265.$$

These two sequences, as well as several other ones derived from them (for example, take all odd numbers above) are not yet in The On-Line Encyclopedia of Integer Sequences. Thus, the problem proposed is (most likely) new. As we will soon witness, it is impossible to fully characterise all such n. In order to do so, one needs to know all primes of the form F^2+1 or $\frac{F^2+1}{2}$. This cannot be done explicitly, even the infinitude of such primes is not known. Summarizing, there exists a plethora of different solutions to the proposed problem. Each of them explicitly describes certain subsequence of (1). However, explicitly or implicitly, any solution is based on the following lemma. It can be proved by a solver rigorously, or just guessed to be true.

Lemma 1. *Suppose, for certain $n \in \mathbb{N}$ there are two consecutive numbers in the set S_n. Then there exist $x, y \in \mathbb{N}$ so that $n = xy(x+1)(y+1)$.*

Proof. Suppose $d_2 < d_1 \le \sqrt{n}$ are such that

$$d_1 + \frac{n}{d_1} + 1 = d_2 + \frac{n}{d_2}. \quad (2)$$

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Let $(d_1, d_2) = y$ (the greatest common divisor), $\frac{d_2}{y} = x$, $\frac{d_1}{y} = z$. Consequently, $(x, z) = 1$, $z > x$. Since $d_2|n$, then $n = xyT$. Now, $d_1|n \Rightarrow zy|xyT \Rightarrow z|xT$. This implies $T = zt$, $t \in \mathbb{N}$. So, $n = xyzt$. In terms of x, z, y, t, equality (2) rewrites as

$$zy + xt + 1 = xy + zt \Rightarrow (z - x)(t - y) = 1.$$

Since we are dealing with integers and $z > x$, this implies $z = x + 1$ and $t = y + 1$. This gives $n = x(x + 1)y(y + 1)$. □

Solution 1 *(just guessing the suitable answer)*. Let

$$n = 6m(m + 1)(2m + 1)(3m + 2), \quad m \in \mathbb{Z} \setminus \{0\}.$$

Let $d_1 = 6m(m + 1)$, $d_2 = 2m(3m + 2)$, $d_3 = 3m(2m + 1)$. These are three divisors of n. Moreover, $d_3 < d_2 < d_1 < \sqrt{n}$. Now,

$$\begin{aligned} d_1 + \frac{n}{d_1} &= 12m^2 + 13m + 2, \\ d_2 + \frac{n}{d_2} &= 12m^2 + 13m + 3, \\ d_3 + \frac{n}{d_3} &= 12m^2 + 13m + 4, \end{aligned}$$

which solves the problem. Values given by this solution in (1) are marked bold. ■

There are many 4th degree polynomials whose values at integers provide the solution. For example, one can take

$$n = 84m(3m + 1)(4m + 1)(7m + 2), \quad m \in \mathbb{Z} \setminus \{0\}.$$

The values produced by this polynomial are underlined in (1). Note that $n = 15120$ is both bold and underlined. In this particular instance, the sets of three consecutive values of $d + \frac{n}{d}$ produced are different, respectively, $\{246, 247, 248\}$, and $\{247, 248, 249\}$. Thus, in total there are four consecutive values in S_n for $n = 15120$:

$$120 + \frac{n}{120} = 246,\ 112 + \frac{n}{112} = 247,\ 108 + \frac{n}{108} = 248,\ 105 + \frac{n}{105} = 249.$$

Such instances of 4 consecutive values are extremely rare, the first such n being those:

$$15.120, 712.800, 3.341.520, 10.533.600, 23.284.800, 85.503.600, 147.026.880, 171.097.920$$

In relation to this, see the last subsection of this solution. Apparently, 5 consecutive values can never occur.

Solution 2. Let $x, y \in \mathbb{N}$, $x \geq y$. Consider the number $n = x(x + 1)y(y + 1)$. It has two divisors $d_1 = xy$, and $d_2 = x(y + 1)$. Now,

$$d_1 + \frac{n}{d_1} = xy + (x + 1)(y + 1) = 2xy + x + y + 1 = 1 + d_2 + \frac{n}{d_2}.$$

In other words, these expressions are two consecutive numbers. We will now look for x, y, D, so that the following equation in positive integers

$$D + \frac{n}{D} = D + \frac{xy(x+1)(y+1)}{D} = 2xy + x + y - 1 \tag{3}$$

has a solution. If this is the case, we witness three consecutive integers among values of $d + \frac{n}{d}$.

The discriminant of the quadratic equation (3) is equal to

$$W = (2xy + x + y - 1)^2 - 4xy(x+1)(y+1) = (x + y - 1)^2 - 8xy.$$

This expression is quadratic in x, y. If we happen to choose x dependent on y so that it factors into two linear expressions, the problem might become easier. It is not difficult to notice that $x = ay + 1$ does the trick, since W then possesses a linear factor y. We can take, for example, $x = 7y + 1$. For this particular choice, $W = 8y(y-1)$. Since y and $y - 1$ are coprime, this is a square of an integer in two cases: either $y = P^2$, $y - 1 = 2Q^2$, $P, Q \in \mathbb{N}_0$, or $y = 2P^2$, $y - 1 = Q^2$, $P, Q \in \mathbb{N}_0$. In the first case we have $P^2 - 2Q^2 = 1$. This is the famous Pell's equation. It is well-known that it has an infinity of solutions, given by $P + Q\sqrt{2} = (3 + 2\sqrt{2})^m$, $m \in \mathbb{N}_0$. Namely, $(P, Q) = (1, 0), (3, 2), (17, 12)$, and so on. Gathering everything in one place, we obtain the following fact. Let (P, Q) is the solution of Pell's equation $P^2 - 2Q^2 = 1$. Then

$$n = P^2(P^2 + 1)(7P^2 + 1)(7P^2 + 2)$$

(first values being $144, 374440, \ldots$) satisfies the conditions of the problem. Thus, there are infinitely many such numbers. ■

Solution 3. First, one notices that an expression (2) gives the following

$$1 < \frac{n}{d_1 d_2} = 1 + \frac{1}{d_1 - d_2}.$$

So, even before starting looking for any suitable construction, one knows beforehand that if there exist three divisors $\leq \sqrt{n}$ which satisfy the condition of the problem, they should be quite close to the upper bound $\sqrt{n}$.

This leads to the following construction. Let $a, b, c, d \in \mathbb{N}$, $x, y, z, w \in \mathbb{Z}$, and

$$n = (am + x)(bm + y)(cm + z)(dm + w) \cdot abcd, \quad m \in \mathbb{Z}.$$

To form three divisors, let

$$\begin{aligned} d_1 &= (am + x)(bm + y)cd, \\ d_2 &= (am + x)(cm + z)bd, \\ d_3 &= (am + x)(dm + w)bc. \end{aligned}$$

A straightforward calculation gives

$$\begin{aligned} d_1 + \frac{n}{d_1} - d_2 - \frac{n}{d_2} &= (bz - cy)(aw - dx), \\ d_2 + \frac{n}{d_2} - d_3 - \frac{n}{d_3} &= (cw - dz)(ay - bx). \end{aligned} \tag{4}$$

If both expressions on the right are equal to $+1$, then n satisfies the condition of the problem for all $m \in \mathbb{Z}$. At first it seems rather tedious to follow all cases where the two factors might be $(+1, +1)$, as well as $(-1, -1)$. In fact, one needs not to worry about the sign. It is enough to achieve the following. Find 4 pairs in the following table:

$$\left|\begin{array}{c|c|c|c} a & b & c & d \\ x & y & z & w \end{array}\right|$$

The only two worries are that columns are not proportional, and "cross-differences" $ay - bx$, $bz - cy$, $cw - dz$, $dx - aw$ are all equal to either $+1$ or -1. Then, in case of necessity renumbering d_1, d_2, d_3, we can always achieve that both expressions in (4) are equal to $+1$. Here are four possible collections:

$$\left|\begin{array}{c|c|c|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right| \qquad \left|\begin{array}{c|c|c|c} 1 & 3 & 7 & 4 \\ 0 & 1 & 2 & 1 \end{array}\right| \qquad \left|\begin{array}{c|c|c|c} 1 & \ell & 2\ell+1 & \ell+1 \\ 0 & 1 & 2 & 1 \end{array}\right| \qquad \left|\begin{array}{c|c|c|c} 2 & 7 & 5 & 3 \\ 1 & 3 & 2 & 1 \end{array}\right|$$

The first two correspond to expressions, respectively, $n = 6m(m+1)(2m+1)(3m+2)$ and $n = 84m(3m+1)(4m+1)(7m+2)$. We had them before. This can be even generalized to a bivariate third expression. Namely,

$$n = \ell(\ell+1)(2\ell+1)m(\ell m+1)\Big(2\ell m + m + 2\Big)\Big(\ell m + m + 1\Big), \ell, m \in \mathbb{Z}.$$

This produces many values in (1), but not all. Fo example, $n = 37800$ is missed.

The last collection correspond to the new solution

$$n = 210(2m+1)(3m+1)(5m+2)(7m+3).$$

General solution. We can describe a complete scenario when three consecutive values among numbers in S_n occur. A direct calculation shows that

$$W = 2(x-y)^2 - (x+y+1)^2 + 2.$$

If $x = y$, then $W < 0$. Therefore, we can assume $x > y$. Let

$$x - y = F, \quad x + y + 1 = G. \tag{5}$$

These are two positive integers of distinct parity. On the other hand, for each pair F, G so that $G \geq F + 3$, $F, G \in \mathbb{N}$, and F, G are of distinct parity, there exists a unique pair $x, y \in \mathbb{N}$ which solves (5). Suppose, we have such a pair (F, G). We want to find all integer solutions to $W = H^2$. In other words,

$$2F^2 + 2 = G^2 + H^2, \text{ where } H \geq 0, \quad G \geq F + 3.$$

For such integer solution, G and F are automatically of distinct parity, and G, H are of the same. Thus,

$$\Big(\frac{G-H}{2}\Big)^2 + \Big(\frac{G+H}{2}\Big)^2 = F^2 + 1.$$

In other words, $F^2+1=A^2+B^2$, $A\geq B$, but $(A,B)\neq(F,1)$. As is known from algebraic number theory, this happens if and only if $2^{-\eta}(F^2+1)$ is not prime; here $\eta=F \pmod 2+1$ (value of mudulo is supposed to be 0 ir 1).

One cannot hope to explicitly describe all prime of the form F^2+1.

1. Calculations

Here is MAPLE code which calculates all values of n satisfying the condition of the problem into a set `Ki`.

First, suppose $n=x(x+1)y(y+1)\leq M$. Then $M>2x^2$. Thus $x<\frac{\sqrt{M}}{\sqrt{2}}$. Further, $G=x+y+1\leq 2x$. Consequently, $F<G<\sqrt{2M}$. Next, one has $x+\frac{1}{2}=\frac{G+F}{2}$, $y+\frac{1}{2}=\frac{G-F}{2}$. Squaring and multiplying, we obtain

$$16\Big(x^2+x+\frac{1}{4}\Big)\Big(y^2+y+\frac{1}{4}\Big)=(G^2-F^2)^2. \tag{6}$$

Since $n=(x^2+x)(y^2+y)$, the left side of (6) is equal to

$$16n+4(x^2+y^2+x+y)+1\leq 16n+4\Big(\frac{n}{2}+2\Big)+1=18n+9<19M.$$

Therefore,

$$G^2-F^2<\sqrt{19M}.$$

Now, $G=F+t$, where $t=2y+1$ is odd. The last inequality gives

$$t^2<Ft+t^2<\sqrt{19M}\Rightarrow t<\sqrt[4]{19M}\Rightarrow y<\frac{1}{2}\sqrt[4]{19M}.$$

So, to calculate all values of the sequence in the range $n\leq M$, we run a double cycle. Variable F runs from 1 to $\sqrt{2M}-4$, while variable $G=F+2y+1$ runs from $y=1$ to $y=\min\big\{\frac{1}{2}\sqrt[4]{19M},\frac{1}{2}(\sqrt{2M}-F)\big\}$.

```
restart;
M:=300000:
Ki:={}: Vi:=floor(sqrt(2*M)):
Ski:=floor((19*M)^(1/4)/2):
for F from 1 to Vi-4 do
for y from 1 to min(floor((Vi-F)/2),Ski) do
G:=F+2*y+1:
if issqr(2*F^2-G^2+2) then
x:=(F+G-1)/2;y:=(G-F-1)/2:
n:=x*(x+1)*y*(y+1):
Ki:=Ki union {n}:
end if:
end do: end do:
Ki;
```

Four consecutive values. To describe instances where four consecutive values in the set S_n occur, one needs to find integer solutions to

$$\begin{aligned} 2(x-y)^2-(x+y+1)^2+2 &= S^2, \\ 3(x-y)^2-2(x+y+1)^2+6 &= T^2. \end{aligned}$$

In other words,

$$2F^2-G^2+2 \text{ and } 3F^2-2G^2+6 \text{ are both squares,}$$

where $G \geq F+3$. One easily adjusts the MAPLE code. There seem to exist infinitely many such n. To find five consecutive values, one also needs to satisfy the condition that $4F^2-3G^2+12$ is also a square. No immediate obstruction is to be seen, but no such values were found.

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