

# THE SIMPLEST ICOSAHEDRAL FLOW ON THE SPHERE

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ABSTRACT. The invariant ring of the rotational icosahedral group is generated by  $\varrho = x^2 + y^2 + z^2$  and by homogeneous invariants  $P_6, P_{10}, P_{15}$ , subject to one weighted degree 30 relation. This algebraic theory goes back to F. Klein. In this paper we show that it has a dynamical counterpart. Namely, consider the flow with the vector field  $F = \nabla \varrho \times \nabla P_6$ . We show that the differential system admits a quotient reduction through curves of genera  $25 \mapsto 7 \mapsto 3 \mapsto 1$ , and the time differential descends onto an elliptic curve defined by  $P_{10}$  and  $P_{15}$ . In particular, on each regular level the flow is integrated by algebraic functions over a field of elliptic functions.

## 1. ICOSAHEDRAL INVARIANTS AND THE FLOW

**Note.** Part of the results presented here (the first two algebraic reductions) appeared in 2017 [1]. The latter manuscript was never submitted, and now serves as a pool for further research. Only recently the author noticed that the third reduction of the differential system is possible. The current paper is thus the focused and finalized version of one chapter in [1].  $\square$

Throughout this paper, we fix  $\phi = \frac{1+\sqrt{5}}{2}$ . Define

$$\alpha \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \beta \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \gamma \mapsto \begin{pmatrix} \frac{1}{2} & -\frac{\phi}{2} & \frac{1}{2\phi} \\ \frac{\phi}{2} & \frac{1}{2\phi} & -\frac{1}{2} \\ \frac{1}{2\phi} & \frac{1}{2} & \frac{\phi}{2} \end{pmatrix}.$$

These are matrices satisfying  $\alpha^2 = \beta^3 = \gamma^5 = I$ . Together they generate orientation-preserving icosahedral group  $\mathbb{I} \subset SO(3)$  of order 60.

In line with the Chevalley-Shephard-Todd theorem the full icosahedral group  $\mathbb{I} \times \{\pm I\}$  of order 120 has a polynomial ring of invariants whose generators can be chosen as follows. First,  $\varrho = x^2 + y^2 + z^2$ . Next,

$$P_6 = (\phi^2 x^2 - y^2)(\phi^2 y^2 - z^2)(\phi^2 z^2 - x^2).$$

A convenient degree-ten invariant is

$$P_{10} = (x^4 + y^4 + z^4 - 2x^2 y^2 - 2x^2 z^2 - 2y^2 z^2) (\phi^2 z^2 - \phi^{-2} y^2) (\phi^2 x^2 - \phi^{-2} z^2) (\phi^2 y^2 - \phi^{-2} x^2).$$

Since  $\mathbb{I}$  is not generated by pseudoreflections, its invariant ring has an additional degree 15 generator. It can be written as follows:

$$P_{15} = xyz \prod_{\varepsilon, \delta = \pm 1} (x + \varepsilon \phi y + \delta \phi^{-1} z) (y + \varepsilon \phi z + \delta \phi^{-1} x) (z + \varepsilon \phi x + \delta \phi^{-1} y).$$

Thus,  $\varrho, P_6, P_{10}$  are primary invariants, and the invariant ring of  $\mathbb{I}$  is a free module of rank 2 over the polynomial ring they generate. The relation between  $\varrho, P_6, P_{10}, P_{15}$  can be written explicitly:

$$\begin{aligned} 6400P_{15}^2 &= 500(\sqrt{5}P_{10})^3 + (80\varrho^5 + 6500\phi^{-3}P_6\varrho^2)(\sqrt{5}P_{10})^2 \\ &\quad + 100\phi^{-3}P_6(720\phi^{-6}P_6^2\varrho + 159\phi^{-3}P_6\varrho^4 + 8\varrho^7)(\sqrt{5}P_{10}) \\ &\quad + 100\phi^{-6}P_6^2(8\phi^{-3}P_6 + \varrho^3)^2(27\phi^{-3}P_6 + 4\varrho^3). \end{aligned}$$

The first non-constant scalar spherical harmonic which contains the trivial representation of  $\mathbb{I}$  occurs in degree  $\ell = 6$  and is equal to

$$H_6 = P_6 + \frac{3 + \sqrt{5}}{7}\varrho^3.$$

Indeed,

$$\Delta P_6 = -6(3 + \sqrt{5})\varrho^2, \quad \Delta(\varrho^3) = 42\varrho^2 \Rightarrow \Delta H_6 = 0.$$

The theory described is classical and goes back to F. Klein. The purpose of this paper is to attach to this theory a canonical dynamical object: the simplest homogeneous polynomial flow on the sphere equivariant under  $\mathbb{I}$ . Here ‘‘simplest’’ has a precise meaning: among homogeneous polynomial vector fields on  $\mathbb{R}^3$  which are equivariant under  $\mathbb{I}$  and tangent to the spheres  $\varrho = \text{const.}$ , the first non-zero degree is 6. Up to scalar multiplication, the corresponding field is

$$F = \left(\frac{5}{2} - \sqrt{5}\right) \nabla \varrho \times \nabla P_6.$$

(If we used  $H_6$  instead of  $P_6$ , the vector field would be the same; to be in line with the vector field used in ([2]), we adopt the normalizing constant as written). Equivalently, in the language of three-dimensional Nambu mechanics, this is the Nambu-Hamiltonian field: if  $\Omega = dx \wedge dy \wedge dz$  is the standard volume form, then  $\iota_F \Omega = (5/2 - \sqrt{5})d\varrho \wedge dP_6$ . From the very construction, the orbits lie on the common level curves of  $\rho$  and  $P_6$ . On the unit sphere  $\varrho = 1$ , the dynamics is the motion along the spherical algebraic curves  $P_6 = \xi$ .

So, the main problem of the current paper is to investigate the differential system  $\dot{\mathbf{x}} = F(\mathbf{x})$ . It can be written explicitly:

$$\begin{cases} p' = -(5 - \sqrt{5})qr^5 - (5 + \sqrt{5})q^5r + 20q^3r^3 - (10 + 10\sqrt{5})p^2qr^3 - (10 - 10\sqrt{5})p^2q^3r + 10p^4qr, \\ q' = -(5 - \sqrt{5})rp^5 - (5 + \sqrt{5})r^5p + 20r^3p^3 - (10 + 10\sqrt{5})q^2rp^3 - (10 - 10\sqrt{5})q^2r^3p + 10q^4rp, \\ r' = -(5 - \sqrt{5})pq^5 - (5 + \sqrt{5})p^5q + 20p^3q^3 - (10 + 10\sqrt{5})r^2pq^3 - (10 - 10\sqrt{5})r^2p^3q + 10r^4pq, \\ p^2 + q^2 + r^2 = 1, \quad (\phi^2p^2 - q^2)(\phi^2q^2 - r^2)(\phi^2r^2 - p^2) = \xi. \end{cases} \quad (1)$$

## 2. THE SPHERICAL LEVEL SETS

The zero set of the sextic invariant on the unit sphere is  $P_6 = 0$ . Equivalently, it is the union of the six great circles

$$\phi x = \pm y, \quad \phi y = \pm z, \quad \phi z = \pm x.$$

These great circles form the separatrix graph of the flow. They divide  $\mathbf{S}^2$  into 32 spherical chambers: 20 triangles and 12 pentagons. The 30 vertices of this graph are saddle equilibria.

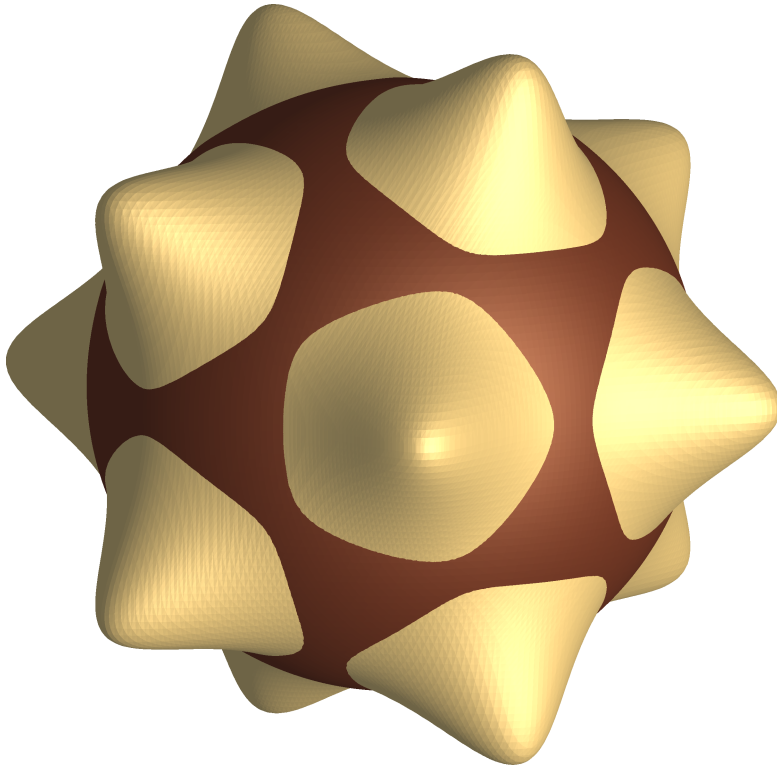


FIGURE 1. The sphere  $\varrho = 1$  with the surface  $P_6 + \varrho^3 = 1 - \frac{1}{25}$ .

The centers of the 20 triangular chambers are maxima of  $P_6|_{\mathbf{S}^2}$ , while the centers of the 12 pentagonal chambers are minima. The range of  $P_6$  on the unit sphere is

$$-\frac{\phi^3}{5} \leq P_6 \leq \frac{\phi^3}{27}.$$

Thus the level sets on  $\mathbf{S}^2$  have the following form. If

$$-\frac{\phi^3}{5} < \xi < 0,$$

then  $P_6 = \xi$  consists of 12 disjoint smooth closed curves, each surrounding one of the minima. If

$$0 < \xi < \frac{\phi^3}{27},$$

then  $P_6 = \xi$  consists of 20 disjoint smooth closed curves, each surrounding one of the maxima. At the endpoint values  $\xi = -\frac{\phi^3}{5}$ ,  $\xi = \frac{\phi^3}{27}$ , the level set collapses respectively to 12 or 20 isolated critical points. At  $\xi = 0$ , the level set is the separatrix graph itself, with 30 vertices and 60 open great-circle arcs.

For visualization it is useful to replace the non-compact surface  $P_6 = \xi$  by the compact star-shaped surface

$$L_\xi = \{P_6 + \rho^3 = 1 + \xi\}.$$

Since  $\rho = 1$  on the unit sphere,

$$L_\xi \cap \mathbf{S}^2 = \{P_6 = \xi\} \cap \mathbf{S}^2.$$

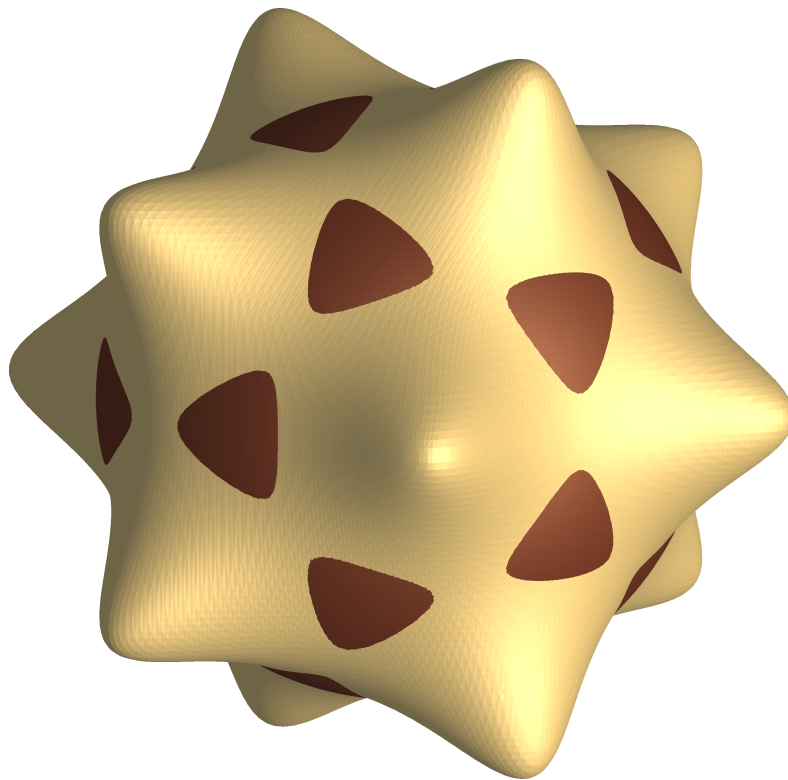


FIGURE 2. The sphere  $\varrho = 1$  with the surface  $P_6 + \varrho^3 = 1 + \frac{1}{25}$ .

Moreover,  $P_6 + \rho^3$  is homogeneous of degree 6. Hence, for  $\xi > -1$ , the surfaces  $L_\xi$  are homothetic, and they provide a convenient compact model for rendering the intersection curves.

Finally, the zero set of the vector field on  $\mathbf{S}^2$  consists of 62 points: the 12 minima, the 20 maxima, and the 30 saddle points of  $P_6|_{\mathbf{S}^2}$ . Up to radial normalization, these are the usual vertices, face-centers, and edge-centers of the icosidodecahedral decomposition determined by the six great circles above.

### 3. THE FIRST REDUCTION

Now, we will make the first simplification to (1). Let

$$P = \phi^2 p^2 - q^2, \quad Q = \phi^2 q^2 - r^2, \quad R = \phi^2 r^2 - p^2. \quad (2)$$

Then  $P + Q + R = \phi$ ,  $PQR = \xi$ . The inverse transformation is

$$\begin{cases} 4\phi p^2 = \phi^2 P + Q + \phi^{-2} R, \\ 4\phi q^2 = \phi^{-2} P + \phi^2 Q + R, \\ 4\phi r^2 = P + \phi^{-2} Q + \phi^2 R. \end{cases} \quad (3)$$

Since  $P = \phi^2 p^2 - q^2$ , then  $P' = 2\phi^2 pp' - 2qq'$ . Substitution of (1) gives

$$P' = (40 - 8\sqrt{5})pqrP(\phi^2 q^2 + p^2 - (\phi^2 + 1)r^2).$$

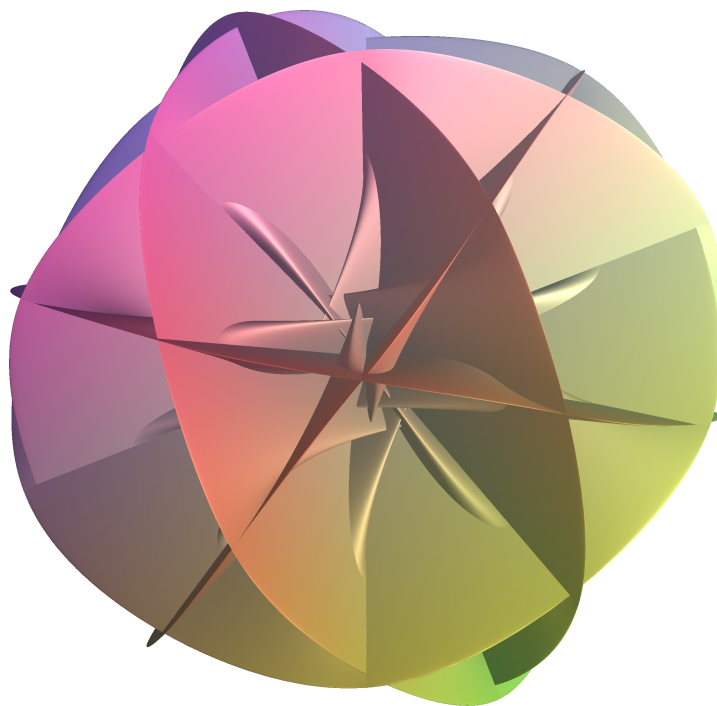


FIGURE 3. The velocity surface, the locus set of  $f : \mathbb{S}^2 \mapsto \mathbb{R}^3$ ,  $f(\mathbf{x}) = F(\mathbf{x})$ .

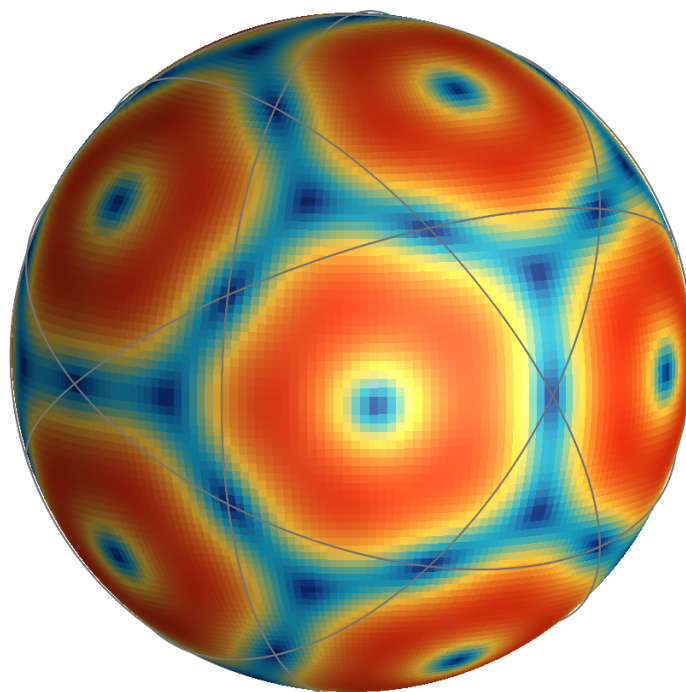


FIGURE 4. Each point of the sphere is colored according to magnitude  $|F|^2$ . Warm colors indicate the biggest speed.

But the factor in parentheses is precisely  $Q - R$ . Squaring and using  $(40 - 8\sqrt{5})^2 = 1280\phi^{-2}$  gives

$$P^2 = 1280\phi^{-2}p^2q^2r^2P^2(Q - R)^2.$$

Since

$$Q + R = \phi - P, \quad QR = \frac{\xi}{P} \quad \Rightarrow (Q - R)^2 = (\phi - P)^2 - \frac{4\xi}{P},$$

and therefore

$$\frac{P'^2}{P^4 - 2\phi P^3 + \phi^2 P^2 - 4\xi P} = 1280\phi^{-2}p^2q^2r^2. \quad (4)$$

Let

$$\mu = P^2Q + Q^2R + R^2P, \quad \nu = PQ^2 + QR^2 + RP^2.$$

Multiplying the three identities in (3) gives

$$64\phi^3p^2q^2r^2 = 22PQR + P^3 + Q^3 + R^3 + \frac{13}{2}(\mu + \nu) + \frac{\sqrt{5}}{2}(\mu - \nu). \quad (5)$$

Using (2),

$$P^3 + Q^3 + R^3 = \phi^3 - 3\phi^2P + 3\phi P^2 - \frac{3\xi\phi}{P} + 3\xi,$$

and

$$\mu + \nu = \phi^2P - \phi P^2 + \frac{\xi\phi}{P} - 3\xi.$$

Substitution into (5) yields

$$128\phi^3p^2q^2r^2 = 11\xi + 2\phi^3 + 7\phi^2P - 7\phi P^2 + \frac{7\xi\phi}{P} + \sqrt{5}(\mu - \nu)$$

Combining (4) and the above, we obtain

$$\frac{P'^2\phi^5}{10(P^4 - 2\phi P^3 + \phi^2 P^2 - 4\xi P)} - 11\xi - 2\phi^3 - 7\phi^2P + 7\phi P^2 - \frac{7\xi\phi}{P} = \sqrt{5}(\mu - \nu). \quad (6)$$

Finally,

$$(\mu - \nu)^2 = (P - Q)^2(Q - R)^2(R - P)^2.$$

Since  $Q, R$  are roots of  $Z^2 - (\phi - P)Z + \frac{\xi}{P} = 0$ , one obtains

$$(\mu - \nu)^2 = \left(2P^2 - \phi P + \frac{\xi}{P}\right)^2 \left(P^2 - 2\phi P + \phi^2 - \frac{4\xi}{P}\right). \quad (7)$$

Define

$$\begin{aligned} f_\xi(X) &= 7\phi X^3 - 7\phi^2 X^2 - (11\xi + 2\phi^3)X - 7\xi\phi, \\ g_\xi(X) &= X^3 - 2\phi X^2 + \phi^2 X - 4\xi, \quad h_\xi(X) = 2X^3 - \phi X^2 + \xi. \end{aligned}$$

**Proposition 1** ( $A_\xi \mapsto B_\xi$  reduction). *Let  $(X, Y) = (P, P')$ ,  $(Q, Q')$ , or  $(R, R')$ . Then*

$$\boxed{X \left( Y^2 \phi^5 + 10 f_\xi(X) g_\xi(X) \right)^2 = 500 h_\xi(X)^2 g_\xi(X)^3.} \quad (8)$$

*Proof.* Equation (8) is obtained by squaring (6), substituting (7), and clearing denominators. The derivation is cyclic in  $P, Q, R$ .  $\square$

The equation (8) defines a curve  $B_\xi$  of genus 7.

## 4. THE SECOND REDUCTION

The equation (8) still describes a chosen root  $P, Q$ , or  $R$ . The next step is to pass to the remaining symmetric coordinate. Let  $Z$  be one of  $P, Q, R$ , and define

$$\Upsilon = \frac{Z^3 - \phi Z^2 - \xi}{Z}. \quad (9)$$

So,  $P, Q, R$  are the roots of  $Z^3 - \phi Z^2 - \Upsilon Z - \xi = 0$ . Thus,  $\Upsilon = -(PQ + QR + RP)$ . Differentiating 9, we obtain

$$\Upsilon' = \frac{h_\xi(Z)}{Z^2} Z'. \quad (10)$$

Put

$$\mathfrak{l}_\xi(X) = 14\phi X - 22\xi - 4\phi^3, \quad \mathfrak{t}_\xi(X) = 20X^3 + 5\phi^2 X^2 - 90\phi\xi X - 135\xi^2 - 20\phi^3\xi.$$

A direct calculation gives the identities

$$Z^4 \mathfrak{l}_\xi \left( \frac{Z^3 - \phi Z^2 - \xi}{Z} \right) \mathfrak{t}_\xi \left( \frac{Z^3 - \phi Z^2 - \xi}{Z} \right) = 10h_\xi(Z)^2 f_\xi(Z) g_\xi(Z), \quad (11)$$

and

$$Z^9 \cdot 4\mathfrak{t}_\xi^3 \left( \frac{Z^3 - \phi Z^2 - \xi}{Z} \right) = 500h_\xi(Z)^6 g_\xi(Z)^3. \quad (12)$$

The second identity follows efficiently from the first and from

$$\mathfrak{l}_\xi \left( \frac{Z^3 - \phi Z^2 - \xi}{Z} \right) = \frac{2f_\xi(Z)}{Z}. \quad (13)$$

**Proposition 2** ( $B_\xi \mapsto C_\xi$  reduction). *Let  $\Upsilon = -(PQ + QR + RP)$ . With  $(X, Y) = (\Upsilon, \Upsilon')$ , one has*

$$\boxed{(Y^2\phi^5 + \mathfrak{l}_\xi(X)\mathfrak{t}_\xi(X))^2 = 4\mathfrak{t}_\xi(X)^3}. \quad (14)$$

*Proof.* Start from (8) for  $Z = P$ , and use (10) to replace  $Z'$  by  $\Upsilon'$ . The identities (11) and (12) transform (8) into (14).  $\square$

The equation (14) defines a curve  $C_\xi$  of genus 3.

## 5. THE THIRD REDUCTION

The final reduction comes from noticing that the 5-fold symmetry of the system (1) was not yet dealt with. Indeed, let us look closely at

$$-\Upsilon = (\phi^2 p^2 - q^2)(\phi^2 q^2 - r^2) + (\phi^2 q^2 - r^2)(\phi^2 r^2 - p^2) + (\phi^2 r^2 - p^2)(\phi^2 p - q^2).$$

Any permutation of  $P, Q, R$  permutes quadratic factors in the above thus leaving  $\Upsilon$  intact. Yet, any equation satisfied by  $\Upsilon(x, y, z)$  is also satisfied by  $\Upsilon(\gamma(x, y, z))$ . This gives a function  $\Upsilon_1$ , which is related to  $\Upsilon = \Upsilon_0$ , but not in a trivial way. Since matrix  $\gamma$  is of order 5, we know *a priori* that the equation (14) is parametrised by 5 pairs  $(\Upsilon_i, \Upsilon'_i)$ ,  $0 \leq i \leq 4$ . These arguments convince us that the 5-fold reduction is still hidden in (14) and needs to be unfolded.

Put

$$U = \frac{Y^2\phi^5 + \mathfrak{l}_\xi(X)\mathfrak{t}_\xi(X)}{2\mathfrak{t}_\xi(X)}. \quad (15)$$

Then the equation of the curve becomes  $U^2 = \mathfrak{t}_\xi(X)$ . Now define

$$J = \frac{X}{2\phi^5} (U + 5\xi - 5\phi X). \quad (16)$$

So, when  $(X, Y) = (\Upsilon, \Upsilon')$ ,  $J$  becomes a function in  $\Upsilon$  and  $\Upsilon'$  (*a posteriori*,  $J$  will turn out to be an elliptic curve). Eliminate  $U$  in the above and square. Formula  $U^2 = \mathfrak{t}_\xi(X)$  gives the quintic equation for  $X$  in terms of  $J$ :

$$X^5 - \phi^2 X^4 - 2\phi\xi X^3 - (\phi^3\xi + 8\xi^2 + \phi^6 J) X^2 + \phi^5\xi JX - \frac{\phi^{10}}{5} J^2 = 0.$$

Now, differentiate (16). Using  $X' = Y$  and  $2UU' = \mathfrak{t}'_\xi(X)Y$ , we get

$$J' = \frac{Y}{2\phi^5} \left( U + 5\xi - 10\phi X + \frac{X\mathfrak{t}'_\xi(X)}{2U} \right) \Rightarrow (J')^2 = \frac{Y^2}{4\phi^{10}} \left( U + 5\xi - 10\phi X + \frac{X\mathfrak{t}'_\xi(X)}{2U} \right)^2.$$

Substitute  $Y^2 = U^2\phi^{-5}(2U - \mathfrak{t}_\xi(X))$  in the above.  $U$  in the denominator factors out, and we get a polynomial in  $U$  and  $X$ . Since  $U^2 = \mathfrak{t}_\xi(X)$ , this polynomial can be transformed into  $U \cdot \Omega_\xi(X) + \Gamma_\xi(X)$ . and (16) gives an elliptic curve

$$D_\xi : (J')^2 = \Gamma_\xi(J),$$

**Proposition 3** ( $C_\xi \mapsto D_\xi$  reduction). *Let  $J$  express in terms of  $(X, Y) = (\Upsilon, \Upsilon')$  by (15) and (16). Then with  $(X, Y) = (J, J')$ , one has*

$$\begin{aligned} Y^2 &= 500X^3 + (80 + 6500\xi\phi^{-3})X^2 + 100\xi\phi^{-3}(720\xi^2\phi^{-6} + 159\xi\phi^{-3} + 8)X \\ &+ 100\xi^2\phi^{-6}(8\xi\phi^{-3} + 1)^2(27\xi\phi^{-3} + 4). \end{aligned}$$

Thus the final map is

$$(X, Y) \longmapsto (J, S),$$

and the time differential descends to  $D_\xi$ :

$$dt = \frac{dJ}{80S}.$$

Finally set

$$\xi = \lambda\phi^3, \quad X = \phi^2x.$$

Then the quintic becomes

$$\boxed{x^5 - x^4 - 2\lambda x^3 - (\lambda + 8\lambda^2 + J)x^2 + \lambda Jx - \frac{J^2}{5} = 0,}$$

and the elliptic quotient becomes

$$D_\lambda : \quad S^2 = \Delta_\lambda(J),$$

where

$$\Delta_\lambda(J) = 500J^3 + (80 + 6500\lambda)J^2 + 100\lambda(720\lambda^2 + 159\lambda + 8)J + 100\lambda^2(8\lambda + 1)^2(27\lambda + 4).$$

Its discriminant is

$$\text{disc}_J(\Delta_\lambda) = -2^{20}5^5\lambda^2(5\lambda + 1)^5(27\lambda - 1)^3.$$

Hence  $D_\lambda$  is nonsingular exactly away from the three exceptional values  $\lambda = -1/5$ ,  $\lambda = 0$ ,  $\lambda = 1/27$ . The  $j$ -invariant is

$$j(D_\lambda) = \frac{256(16875\lambda^3 - 2875\lambda^2 + 25\lambda - 1)^3}{3125\lambda^2(5\lambda + 1)^5(27\lambda - 1)^3}.$$

## 6. THE QUOTIENT PICTURE FROM ABOVE

The reductions above were obtained by explicit elimination. We now explain why they are natural from the invariant-theoretic point of view. All quotients below are understood after normalization. Let

$$A_\xi: \quad x^2 + y^2 + z^2 = w^2, \quad P_6(x, y, z) = \xi w^6$$

be the projective orbit curve. For generic  $\xi$ , this is a smooth complete intersection of type  $(2, 6)$  in  $\mathbb{P}^3$ , hence

$$g(A_\xi) = 1 + \frac{2 \cdot 6(2 + 6 - 4)}{2} = 25.$$

The group  $\mathbb{I}$  acts on  $A_\xi$ . The first two reductions are explained by the subgroup chain  $V_4 \subset A_4 \subset A_5$  (it is standard that  $\mathbb{I} \cong A_5$ , the alternating group). Indeed, write

$$P = \phi^2 x^2 - y^2, \quad Q = \phi^2 y^2 - z^2, \quad R = \phi^2 z^2 - x^2$$

in the affine chart  $w = 1$ . On  $A_\xi$  one has  $P + Q + R = \phi$ ,  $PQR = \xi$ . The subgroup  $H \simeq V_4$  of even sign changes fixes  $P, Q, R$  individually. Thus the curve  $B_\xi$ , parametrized by any of  $(P, P')$ ,  $(Q, Q')$ ,  $(R, R')$ , is  $A_\xi/H$ . For generic  $\xi$  this quotient is unramified, and Riemann–Hurwitz gives

$$2g(A_\xi) - 2 = 4(2g(B_\xi) - 2),$$

so  $g(B_\xi) = 7$ . The normalizer of  $H$  in  $G$  is a tetrahedral subgroup  $K = N_G(H) \simeq A_4$ , and  $K/H \simeq C_3$  cyclically permutes  $P, Q, R$ . Therefore the passage from a chosen factor to the symmetric coordinate  $\Upsilon = -(PQ + QR + RP)$  is precisely the quotient by this order-three symmetry. Hence  $C_\xi \simeq A_\xi/K$ . Again, for generic  $\xi$ ,

$$2g(B_\xi) - 2 = 3(2g(C_\xi) - 2),$$

and therefore  $g(C_\xi) = 3$ . Thus the first part of the computation is the quotient chain

$$A_\xi \longrightarrow B_\xi \longrightarrow C_\xi, \quad 25 \longrightarrow 7 \longrightarrow 3.$$

The final curve  $D_\xi$  has an even simpler conceptual origin. On  $A_\xi$ , the two icosahedral invariants  $\varrho, P_6$  are fixed. Therefore the full quotient  $A_\xi/G$  is described by the remaining two invariant coordinates  $P_{10}$  and  $P_{15}$ , satisfying

$$\left(\frac{P_{15}}{w^{15}}\right)^2 = R_{30}\left(1, \xi, \frac{P_{10}}{w^{10}}\right).$$

For generic  $\xi$ , this is an elliptic curve. This curve is exactly  $D_\xi = A_\xi/G$ .

The normalization used in the preceding section is adapted to the time parametrization. Put

$$J = \sqrt{5} \frac{P_{10}}{w^{10}}, \quad \Sigma = J'.$$

With our normalization of the vector field one has

$$J' = 80 \frac{P_{15}}{w^{15}}.$$

Hence the full quotient can be written in the time-normalized form

$$D_\xi : \quad \Sigma^2 = \Gamma_\xi(J),$$

where  $\Gamma_\xi$  is the cubic obtained in the previous section. Equivalently,

$$\Gamma_\xi(J) = 6400 R_{30} \left( 1, \xi, \frac{J}{\sqrt{5}} \right).$$

Thus the elliptic curve found by elimination is not accidental: it is simply the full invariant quotient of  $A_\xi$ .

Since

$$C_\xi \simeq A_\xi/A_4, \quad D_\xi \simeq A_\xi/A_5,$$

there is a natural map

$$C_\xi \longrightarrow D_\xi$$

of degree

$$[A_5 : A_4] = 5.$$

This is the intrinsic meaning of the hidden fivefold structure. It is not, in general, an automorphism of  $C_\xi$ ; it is a five-sheeted map from the tetrahedral quotient to the full icosahedral quotient. In terms of function fields,

$$\mathbb{C}(A_\xi)^{A_5} \subset \mathbb{C}(A_\xi)^{A_4}.$$

The five conjugate quantities  $\Upsilon_i$  are the five branches of this cover. The quintic equation for  $X = \Upsilon$  over  $J$ , found above by calculation, is therefore the equation of the cover

$$A_\xi/A_4 \longrightarrow A_\xi/A_5.$$

Riemann–Hurwitz also reflects this picture. The quotient  $A_\xi \rightarrow C_\xi$  has degree 12 and is generically unramified:

$$2g(A_\xi) - 2 = 12(2g(C_\xi) - 2).$$

The degree-five map  $C_\xi \rightarrow D_\xi$  has target genus 1. Therefore

$$2g(C_\xi) - 2 = 5(2g(D_\xi) - 2) + R$$

gives total ramification

$$R = 4.$$

Thus the last cover is the only place where ramification remains in the generic quotient chain.

Finally, the time differential descends to  $D_\xi$ . On  $C_\xi$  we have  $dt = dX/Y$ , while on  $D_\xi$ , by definition of  $\Sigma$ ,  $dt = dJ/\Sigma$ . Therefore the periods of the flow are elliptic periods on the full icosahedral quotient  $D_\xi$ . The genus-three curve  $C_\xi$  is needed not for the clock, but for the algebraic reconstruction of the reduced coordinate  $\Upsilon$ . Then  $P, Q, R$ , and finally the original flow variables, are obtained by further algebraic operations. After the normalization  $\xi = \lambda\phi^3$ , we write the final quotient as

$$D_\lambda : \quad \Sigma^2 = \Delta_\lambda(J), \quad \Delta_\lambda(J) = \Gamma_{\lambda\phi^3}(J).$$

The exceptional values

$$\lambda = -\frac{1}{5}, \quad \lambda = 0, \quad \lambda = \frac{1}{27}$$

are precisely the critical levels of  $P_6$  on the sphere. Away from them,  $D_\lambda$  is elliptic, and the icosahedral flow is integrated by algebraic functions over the elliptic function field of  $D_\lambda$ .

### 7. THE SPECIAL CASE $\lambda = 1/35$

To illustrate the whole reduction, we will work on the case  $\lambda = \frac{1}{35}$ ,  $\xi = \frac{\phi^3}{35}$ . The real level on the unit sphere thus consists of 20 congruent closed orbits, each surrounding one of the triangular centres. For this value, put

$$T(X) = 4900X^3 + 1225X^2 - 630X - 167.$$

Let  $x_1 < x_2 < x_3$  be the three real roots of  $T(X) = 0$ . Numerically,

$$x_1 \approx -0.311628793229042, \quad x_2 \approx -0.301323600531269, \quad x_3 \approx 0.362952393760311.$$

On the interval  $[x_1, x_2]$ , one has  $t(X) > 0$ . The return time of one full positive orbit can be written as the bielliptic period

$$t_0 = \int_{x_1}^{x_2} \frac{105\sqrt{7}}{\sqrt{T(x)}} \left( \frac{1}{\sqrt{162 - 490x + 2\sqrt{5}\sqrt{T(x)}}} + \frac{1}{\sqrt{162 - 490x - 2\sqrt{5}\sqrt{T(x)}}} \right) dx.$$

Numerically  $t_0 = 1.72498412245_+$ . We now pass to the full icosahedral quotient  $D_{1/35}$ . In this case,

$$\Delta_{1/35}(J) = 500J^3 + \frac{1860}{7}J^2 + \frac{12868}{343}J + \frac{1235132}{2100875}.$$

The change of variables

$$\mathcal{X} = \frac{6125J + 1085}{64}, \quad \mathcal{Y} = \frac{42875}{1024}\Sigma$$

puts this elliptic curve into the simple Weierstrass form

$$\mathcal{Y}^2 = \mathcal{X}^3 - 175\mathcal{X} - 875.$$

Equivalently, after the scaling

$$\mathcal{X} = 5Z, \quad \mathcal{Y} = 5\sqrt{5}Z,$$

we obtain<sup>1</sup>

$$Z^2 = Z^3 - 7Z - 7.$$

Skipping all the details, here is the final result. Let

$$z_1 < z_2 < z_3$$

be the three real roots of  $Z^3 - 7Z - 7 = 0$ . Numerically,

$$z_1 = -1.692021471630096_+, \quad z_2 = -1.356895867892209_+, \quad z_3 = 3.048917339522305_+.$$

The same return time can be written as the elliptic period

$$t_0 = \frac{21}{8\sqrt{5}} \int_{z_1}^{z_2} \frac{dz}{\sqrt{z^3 - 7z - 7}} = 1.72498412245_+.$$

<sup>1</sup>Surely, the showcase level  $\lambda$  was chosen in advance in order this form to be simple.

Skipping all the details (performed by computer algebra engine), here is the final result.

**Proposition 4.** *Let  $\xi = \frac{\phi^3}{35}$ , and  $\wp$  be the Weierstrass elliptic function with the equation*

$$(\wp'(z))^2 = 4\wp(z)^3 - \frac{7}{4}\wp(z) - \frac{7}{16}.$$

*Then, for  $(p, q, r)$  being the solution of (1), the function  $p = p(t)$  satisfies*

$$64\wp^3 + 112p^2 A(p^2)\wp^2 + 28B(p^2)\wp - 7C(p^2) = 0,$$

*where*

$$A(\omega) = 69888\omega^4 - 48720\omega^3 + 12005\omega^2 - 1125\omega + 25,$$

$$B(\omega) = 219520000\omega^{10} - 510384000\omega^9 + 477456000\omega^8 - 236454400\omega^7 \\ + 68247200\omega^6 - 11753728\omega^5 + 1169840\omega^4 - 58940\omega^3 + 700\omega^2 + 50\omega - 1,$$

$$C(\omega) = 215129600000\omega^{15} - 1075648000000\omega^{14} + 2366425600000\omega^{13} - 3027180800000\omega^{12} \\ + 2507796480000\omega^{11} - 1418318720000\omega^{10} + 562256576000\omega^9 - 157725120000\omega^8 \\ + 31164156800\omega^7 - 4251553600\omega^6 + 384718208\omega^5 - 21512960\omega^4 + 659820\omega^3 - 9100\omega^2 + 1.$$

*and*

$$\wp = \wp\left(\frac{16\sqrt{5}}{7}t + z_0\right).$$

This gives an explicit algebraic relation between a coordinate of the original icosahedral flow and one fixed Weierstrass elliptic function.

## REFERENCES

- [1] G. ALKAUSKAS, *Projective superflows. II.  $O(3)$  and the icosahedral group*, 2017 arXiv:1606.05772.
- [2] G. ALKAUSKAS, Beltrami vector field with an icosahedral symmetry, <https://arxiv.org/abs/1706.09295>.
- [3] C. CHEVALLEY, Invariants of finite groups generated by reflections, *Amer. J. Math.* **77** (1955), 778–782.
- [4] D.J. BENSON, *Polynomial invariants of finite groups*. London Mathematical Society Lecture Note Series, 190. Cambridge University Press, Cambridge, (1993).
- [5] N. V. COHAN, *The spherical harmonics with the symmetry of the icosahedral group*, Mathematical Proceedings of the Cambridge Philosophical Society **54** (1958), 28–38.
- [6] H.S.M. COXETER, *Regular polytopes*. Third edition. Dover Publications, Inc., New York, 1973.
- [7] C.J. CUMMINS, J. PATERA, Polynomial icosahedral invariants. *J. Math. Phys.* **29**(8) (1988), 1736–1745
- [8] I. DOLGACHEV, Quartic surfaces with icosahedral symmetry, <http://arxiv.org/abs/1604.02567>.
- [9] P. M. GADEA J. MUÑOZ MASQUÉ, I. V. MYKYTYUK, *Analysis and algebra on differentiable manifolds. A workbook for students and teachers*. Second edition. Problem Books in Mathematics, Springer, London, 2013.
- [10] J. P. JOUANOLOU, *Equations de Pfaff algébriques*, Lect. Notes in Math. **708**, Springer-Verlag, Berlin (1979)
- [11] M. Field, *Dynamics and Symmetry*, Imperial College Press Advanced Texts in Mathematics, Vol. 3, Imperial College Press, London, 2007.
- [12] M. KONTSEVICH, D. ZAGIER, Periods, *Mathematics unlimited - 2001 and beyond*, Springer, Berlin, 2001, 771–808.
- [13] F. Klein, *Lectures on the Icosahedron and the Solution of Equations of the Fifth Degree*, Trübner, London, 1888.

- [14] J. Llibre and A. C. Murza, Darboux theory of integrability for real polynomial vector fields on  $S^n$ , *Dynamical Systems* **33** (2018), 1–14.
- [15] B. Meyer, *On the symmetries of spherical harmonics*, Canadian Journal of Mathematics **6** (1954), 135–157.
- [16] J. MILNE, *Algebraic geometry*, v6.00, <http://www.jmilne.org/math/CourseNotes/ag.html>.
- [17] I. NIKOLAEV, E. ZHUZHOMA, *Flows on 2-dimensional manifolds*. An overview. Lecture Notes in Mathematics, 1705. Springer-Verlag, Berlin (1999).
- [18] J. V. PEREIRA, Vector fields, invariant varieties and linear systems, *Ann. Inst. Fourier (Grenoble)*, **51**(5) (2001), 1385–1405.
- [19] G. C. SHEPHARD, J. A. TODD, Finite unitary reflection groups, *Canadian J. Math.* **6**, (1954) 274–304.
- [20] L. SMITH, *Polynomial invariants of finite groups*. Research Notes in Mathematics, 6. A K Peters, Ltd., Wellesley, MA, 1995.
- [21] J.K. VERMA, Ring of invariants of finite groups, <http://www.math.iitb.ac.in/~jkv/invar.pdf>
- [22] M. YAMAZAKI, MASAO, Two-dimensional stationary Navier-Stokes equations with 4-cyclic symmetry. *Math. Nachr.* **289**(17-18) (2016), 2281–2311.
- [23] F. KLEIN, *Lectures on the Icosahedron and the Solution of Equations of the Fifth Degree*. Translated by G. G. Morrice, Dover Publications, New York, 1956.
- [24] P. DOYLE AND C. MCMULLEN, Solving the quintic by iteration. *Acta Math.* **163** (1989), 151–180.
- [25] O. NASH, On Klein’s icosahedral solution of the quintic. *Expo. Math.* **32**(2) (2014), 99–120.
- [26] A. NAGANO, Icosahedral invariants and a construction of class fields via periods of (K3) surfaces. *Ramanujan J.* **46**(1) (2018), 201–227.
- [27] A. NAGANO, Icosahedral invariants and Shimura curves. *J. Th’eor. Nombres Bordeaux* **29**(2) (2017), 603–635.
- [28] J. STEVENS, Semistable (K3)-surfaces with icosahedral symmetry. *Enseign. Math.* **48** (2002), 91–126.
- [29] YU. G. PROKHOROV, Icosahedron in birational geometry. Preprint, arXiv:2411.15334 (2024).

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