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FUNCTIONAL APPROACH IN THE DIVISOR DISTRIBUTION PROBLEMS

E. MANSTAVIČIUS* (Vilnius)

1. Results

One-dimensional laws of the iterated logarithm for additive arithmetic functions have been investigated in [1], [9] and in several papers by the author. Our main results were summarized in [13]. Independently, R. R. Hall and G. Tenenbaum [7] considered a partial case and gave a new number-theoretical application. On the other hand, the functional limit theorems for arithmetical processes have a fairly large literature too. We mention here only [14] and [20] containing the prehistory of the subject. In the present remark based upon the Kubilius probabilistic approach [9] we present the functional law of iterated logarithm of Strassen type (see [10], [19], and [21]).

Let us observe one inconvenient feature of the theorems concerning the strong convergence of sequences of arithmetical functions or arithmetical processes. Usually, we have a sequence of probabilistic spaces which is difficult to inbed into some fixed and at the same time not too complicated space. As far as we know, the authors who have dealt with the law of iterated logarithm for sums of independent random variables (i.r.v.'s), except W.Feller [5] did not pay any attention to the origin of the common probability space. In some number theoretical problems, contrary to our case (see, for instance, [17]) the product space is quite appropriate. To get over this obstacle, we [11] have suggested a new approach.

Our concept is based on the following definitions. Let $(\mathbf{S}, d)$ be a separable metric space and $Y, Y_1, \dots, Y_n$ be $\mathbf{S}$-valued random elements all defined on the probability space $\{\Omega_n, \mathcal{F}_n, P_n\}$, $n = 1, 2, \dots$.

Let, as usual,

$$d(X,A) = \inf\{\, d(X,Z) : Z \in A\}, \qquad A \subset \mathbf{S}.$$

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We say that Y_k converges to Y P_n-almost surely (P_n-a.s.) if for each $\varepsilon > 0$

$$\lim_{x\to\infty} \limsup_{n\to\infty} P_n\Big(\max_{x \leqq k \leqq n} d(Y_k, Y) \geqq \varepsilon\Big) = 0$$

Thus, a compact set $A \subset \mathbf{S}$ such that, for each $\varepsilon > 0$ and each $X \in A$,

$$\lim_{x\to\infty} \limsup_{n\to\infty} P_n\Big(\max_{x \leqq k \leqq n} d(Y_k, A) \geqq \varepsilon\Big) = 0 \tag{1}$$

and

$$\lim_{x\to\infty} \liminf_{n\to\infty} P_n\Big(\min_{x \leqq k \leqq n} d(Y_k, X) < \varepsilon\Big) = 1 \tag{2}$$

may be called a cluster set of the sequence $\{Y_k\}$ P_n-a.s. In what follows we denote the relations (1) and (2) by

$$Y_k \Longrightarrow A \qquad (P_n\text{-a.s.}). \tag{3}$$

Instead of (2) sometimes we shall write

$$Y_{k_1} \longrightarrow X \qquad (P_n\text{-a.s.})$$

having in mind that k_1 is a random increasing subsequence. The following lemma will be repeatedly used to deduce corollaries from the principal result.

LEMMA 1. *Let* $(\mathbf{S}, d)$ *and* $(\mathbf{S}_1, d_1)$ *be separable metric spaces and let* $f : \mathbf{S} \to \mathbf{S}_1$ *be a continuous map into* $\mathbf{S}_1$. *Then* (3) *in* $(\mathbf{S}, d)$ *implies the convergence*

$$f(Y_k) \Longrightarrow f(A) \qquad (P_n\text{-}a.s.)$$

in the second space $(\mathbf{S}_1, d_1)$.

The proof is evident (see [6], Lemma 137, or [21], Lemma 1.5.11).

If $P_n = P$ does not depend on n, due to monotonicity of the events in (1) and (2), one returns to the traditionally treated situation. Here it should be observed that the proposition $Y_k \Longrightarrow A$ for a compact set A is equivalent to two assertions: the sequence Y_k is relatively compact and A is the set of its limit points. Observe that in the concept of strong convergence introduced above we need the separability of the spaces to assure the measurability of the distances only. This condition is superfluous when P_n has a finite carrier, as in the case considered afterwards. So, at the cost of simplicity we have gained the desired property: we do not need a product space or some other construction of the fixed probabilistic space, common for all Y_k, $k \geqq 1$.

In the present paper we consider sequences of functions related to the multiplicative structure of natural numbers. Let $\mathbf{C} = \mathbf{C}[0,1]$ be the Banach space of continuous functions on the interval $[0,1]$ endowed with the supremum distance $\rho(\cdot,\cdot)$. We recall that the Strassen set $\mathcal{K}$ agrees with the set of absolutely continuous functions g such that $g(0)=0$ and

$$\int_0^1 \left(g'(t)\right)^2 dt \leqq 1.$$

Now we return to arithmetical functions. Let

$$h_k(m) = \sum_{\substack{p^\alpha \| m \\ p \leqq k}} h(p^\alpha),$$

where $h(p^\alpha) \in \mathbf{R}$, $m, \alpha \in \mathbf{N}$, and p stands for a prime number. As usual, $p^\alpha \parallel m$ denotes that p^α divides m but $p^{\alpha+1}$ does not. We put

$$A(u) = \sum_{p \leqq u} \frac{h(p)}{p}, \quad D(u) = \sum_{p \leqq u} \frac{h^2(p)}{p}\left(1 - \frac{1}{p}\right), \text{ and } \beta(u) = \sqrt{2D(u)L_2D(u)},$$

where $Lv = \log\max\{v, e\}$ and $L_k u = L(L_{k-1}u)$, $k \geqq 1$. Let q, q' be two consecutive primes belonging to the set $S(h) = \{p : h(p) \neq 0\}$. For a fixed natural number m, joining the points $(0,0)$ and $\big(D(q), h_q(m) - A(q)\big)$, $q \in \in S(h)$, by straight lines in the coordinate plane we get the graph of the function defined by

$$H(m,t) = \big(h_q(m) - A(q)\big)\frac{D(q') - t}{D(q') - D(q)} + \big(h_{q'}(m) - A(q')\big)\frac{t - D(q)}{D(q') - D(q)}$$

when $D(q) \leqq t < D(q')$. Let further $G_k(m,t) = \beta(k)^{-1}H(m, D(k)t)$ when $0 \leqq t \leqq 1$ and $k \geqq k_0 > q_0 = \min\big\{q : q \in S(h)\big\}$. Thus, $G_k(m,\cdot) \in \mathbf{C}$. Observe that we use natural subscripts k for the sequence $G_k(m,t)$ instead of primes q for convenience only.

Supposing that each $m \leqq n$ is taken at random with equal probability $1/n$, we insert the sequence $G_k(m,\cdot)$, $k \geqq k_0$, in the context described above. Formally speaking one can take $\Omega_n = \mathbf{N}$, $\mathcal{F}_n = 2^{\mathbf{N}}$, and $P_n = \nu_n$ where

$$\nu_n(A) = n^{-1}\#\{m \leqq n, m \in A\}, \qquad A \subset \mathbf{N}.$$

Our main result is the following theorem.

THEOREM 1. *Let* $D(p) \to \infty$ *and*

$$h(p) = o\left(\sqrt{\frac{D(p)}{L_2\, D(p)}}\right) \tag{4}$$

as $p \to \infty$. *Then we have*

$$G_k(m,\cdot) \Longrightarrow \mathcal{K} \qquad (\nu_n\text{-a.s.}) \tag{5}$$

in the space **C**.

Applying Lemma 1, from Theorem 1 one can derive some of the known results as well as new ones. For instance, we have

COROLLARY 1 ([12]). *If the conditions of Theorem* 1 *are satisfied, then*

$$G_k(m,1) \Longrightarrow [-1,1] \qquad (\nu_n\text{-a.s.}).$$

COROLLARY 2. *If the conditions of Theorem* 1 *are satisfied, then*

$$\Big(G_k(m,1/2), G_k(m,1)\Big) \Longrightarrow \mathcal{L} := \Big\{(u,v) : u^2 + (v-u)^2 \leqq 1/2\Big\} \quad (\nu_n\text{-a.s.}).$$

More sophisticated applications can be based on the ideas of [6] and [21]. Further we shall present several examples characterizing features of the last relation.

Theorem 1 has its analogue in the space $\mathbf{D} = \mathbf{D}[0,1]$ of right continuous functions defined on the unit interval and having limits from the left. In our case instead of the Skorokhod distance in **D** we can use the metrics ρ.

THEOREM 2. *Let the sequence of step functions* $U_k(m,t)$ *be defined by*

$$U_k(m,t) = \beta(k)^{-1}\big(h_q(m) - A(q)\big)$$

when $D(q)/D(k) \leqq t < D(q')/D(k)$ *and* $q \leqq k$. *If the conditions of Theorem* 1 *are satisfied, then*

$$U_k(m,\cdot) \Longrightarrow \mathcal{K} \qquad (\nu_n\text{-a.s.})$$

in the space **D**.

If $\omega(m)$ denotes the number of different prime divisors of $m \in \mathbf{N}$ and $\omega(m,u) = \omega_u(m)$, then we have $A(x) \sim D(x) = L_2x + O(1)$ as $x \to \infty$. Thus, setting $b(k) := \big(2(L_2k)L_4k\big)^{1/2}$,

$$\Psi_k(m,t) = \frac{\omega\left(m, \exp\left\{(Lk)^t\right\}\right) - tL_2k}{b(k)}$$

we obtain

COROLLARY 1. *The following relations hold (ν_n-a.s.):*
i) $\Psi_k(m,\cdot) \Longrightarrow \mathcal{K}$;
ii) $\Psi_k(m,1) \Longrightarrow [-1,1]$;
iii) $\big(\Psi_k(m,1/2),\Psi_k(m,1)\big) \Longrightarrow \mathcal{L}$;
iv) $\Psi_k(m,1/2) \Longrightarrow [-\sqrt{2}/2,\sqrt{2}/2]$;
v) *if* k_1 *is the subsequence for which* $\Psi_{k_1}(m,1/2) \to \sqrt{2}/2$, *then we have* $\Psi_{k_1}(m,\cdot) \to g_1$, *where*

$$g_1(t) = \begin{cases} t\sqrt{2}, & \text{if} \quad 0 \leqq t \leqq 1/2, \\ \sqrt{2}/2, & \text{if} \quad 1/2 \leqq t \leqq 1; \end{cases}$$

vi) *if* k_1 *is the subsequence for which* $\Psi_{k_1}(m,1/2) \to 1/2$ *and* $\Psi_{k_1}(m,1) \to$ $\to 0$, *then we have* $\Psi_{k_1}(m,t) \to g_2$, *where*

$$g_2(t) = \begin{cases} t, & \text{if} \quad 0 \leqq t \leqq 1/2, \\ 1-t, & \text{if} \quad 1/2 \leqq t \leqq 1. \end{cases}$$

The proof is based upon the calculations of all possible values on $g \in \mathcal{K}$ of related functionals. Details are given in [6] (pp. 87, 88) but the evident misprint in the definition of the function $g(x)$ on line 8 of page 88 should be corrected.

In 1969 P.Erdős [2] proved that

$$\omega(m,t) - \omega(m,s) \sim L\frac{Lt}{Ls}$$

for almost all m uniformly in $s < t \leqq m$ provided that

$$Lt \geqq (Ls)\exp\{\psi(m)L_3m\}$$

with arbitrary $\psi(m) \to \infty$ as $m \to \infty$. In particular case we can say more.

COROLLARY 2. *We have*

$$\Psi_k(m,1) - \Psi_k(m,1/2) \Longrightarrow \left[-\sqrt{2}/2,\sqrt{2}/2\right] \qquad (\nu_n\text{-a.s.}).$$

PROOF. This assertion is contained in (iii) of Corollary 2.

Let us consider another example (see [7])

$$\Delta_k(m,t) = \frac{\log_2 \delta\left(m, \exp\left\{(Lk)^t\right\}\right) - tL_2k}{b(k)},$$

where $\delta(m,u) = \#\{d \in \mathbf{N} : d|m, p(d) \leqq u\}$. Here $p(d)$ stands for the maximal prime divisor of d.

COROLLARY 3. *All assertions in Corollary* 1 *to Theorem* 2 *remain valid if one replaces* $\Psi_k(m,\cdot)$ *by* $\Delta_k(m,\cdot)$.

In [7] on pages 25, 26 a proof of the one dimensional law of iterated logarithm for the function $\tau(m,t) := \#\{d \in \mathbf{N} : d|m, d \leqq t\}$ has been sketched. We shall obtain the functional form of the law.

THEOREM 3. *Let*

$$T_k(m,t) = \frac{\log_2 \tau\left(m, \exp\left\{(Lk)^t\right\}\right) - tL_2k}{b(k)}, \qquad 0 \leqq t \leqq 1.$$

Then $T_k(m,\cdot) \Longrightarrow \mathcal{K} \quad (\nu_n\text{-}a.s.)$.

Further, we shall deal with the inverses of functions and sequences involved too. Proposition (ii) of Corollary 1 implies that

$$\max\left\{\left|\frac{\omega(m,k) - L_2k}{b(k)}\right| : x \leqq k \leqq n\right\} \leqq 1 + \varepsilon$$

for each $\varepsilon > 0$ and almost all m provided that n and x are sufficiently large. Let $p_j(m)$, $1 \leqq j \leqq \omega(m)$, be the j-th prime factor of m. It can be called the success epoch of the function $\omega(m,\cdot)$ (compare with that treated in [21]). Then the last inequality with the choice $k = p_j(m)$ yields

$$\text{(E)} \qquad \left|L_2p_j(m) - j\right| \leqq (1+\varepsilon)\sqrt{2jL_2j}$$

uniformly in $x(m) \leqq j \leqq \omega(m)$ for almost all m provided that $x(n) \to \infty$ as $n \to \infty$. Similarly, starting from the estimates given on page 292 in [13] one obtains the following sharper inequality

$$|L_2p_j(m) - j| \leqq \left(2j(L_2j + \frac{3}{2}L_3j + L_4j + \ldots + (1+\varepsilon)L_sj\right)^{1/2},$$

valid for each $s \geqq 4$. These estimates considered for all j, $x(m) \leqq j \leqq \omega(m)$, with sufficiently slowly increasing $x(m)$ can not be essentially improved, that is, one can not substitute $1-\varepsilon$ in place of $1+\varepsilon$.

The estimate (E) has been announced in 1946 by P. Erdős [1] (see [2] and [3] too). It is proved in the monograph [7] (Theorem 12), where further very deep applications are presented. Later the investigation of the sequence $p_j(m)$ was continued in [4] and [14]. Our functional approach sheds more light on the problem.

Denote $p(m,u) = p_{[u]}(m)$. Here and in what follows the domain for u is to be confined to $0 \leqq u \leqq \omega(m)$ and we put $p(m,u) = 0$ when $0 \leqq u < 1$. Let

$$P_k(m,t) = \frac{L_2p(m,tk) - tk}{\sqrt{2kL_2k}}, \qquad t \in [0,1].$$

The following law of iterated logarithm is true.

THEOREM 4. *We have*

$$P_k(m,\cdot) \Longrightarrow \mathcal{K} \qquad (\nu_n\text{-a.s.}).$$

The last assertion implies

COROLLARY. *We have*

$$P_k(m,1) \Longrightarrow [-1,1] \qquad (\nu_n\text{-a.s.})$$

and

$$\big(P_k(m,1/2), P_k(m,1)\big) \Longrightarrow \mathcal{L}.$$

Other propositions mentioned in Corollaries of Theorem 2 in the case of $P_k(m,t)$ hold true too.

Inverting Theorem 3 we obtain a result about $d(m,u)$, the $[u]$-th natural divisor of $m \in \mathbf{N}$ with the convention that $d(m,u)=0$ when $0 \leqq u < 1$.

THEOREM 5. *Let*

$$D_k(m,t) := \frac{L_2 d(m,k^t) - t(Lk)/L2}{\sqrt{2\big((Lk)L_3k\big)/L2}}, \qquad t \in [0,1].$$

Then $D_k(m,\cdot) \Longrightarrow \mathcal{K} \quad (\nu_n\text{-a.s.})$.

COROLLARY. *All assertions in Corollary* 1 *to Theorem* 2 *remain valid if one replaces* $\Psi_k(m,\cdot)$ *by* $D_k(m,\cdot)$.

Observe that this corollary extends Theorem 13 of [7].

Probability theory suggests two equivalent approaches to deal with $p(m,u)$ or $d(m,u)$. The first one, described in [8], uses the concept of counting processes, while the second one, considered in [21], is based on generalized inverses of nondecreasing functions. We exploit the ideas of both sources.

Direct theorems

We divide the proof into several simple auxiliary lemmas. In limiting passages we always take at first $n \to \infty$ and then $x \to \infty$.

Let $\hat{h}_k(m)$ and $\widehat{G}_k(m,t)$ be the functions obtained from $h_k(m)$ and $G_k(m,t)$ using the numbers $\hat{h}(p^\alpha) = h(p)$ instead of $h(p^\alpha)$ for all $\alpha \geqq 1$ and primes p.

LEMMA 2. *For each* $\varepsilon > 0$,

$$\lim_{x\to\infty} \limsup_{n\to\infty} \nu_n\left(\max_{x\leqq k\leqq n} \rho\big(G_k(m,\cdot), \widehat{G}_k(m,\cdot)\big) \geqq \varepsilon \right) = 0. \tag{7}$$

PROOF. Since for $q, q' \in S(h)$

$$\max\left\{ \left| \widehat{G}_k(m,t) - G_k(m,t) \right| : D(q)/D(k) \leqq t \leqq D(q')/D(k) \right\} \leqq$$
$$\leqq \beta(k)^{-1} \max\left\{ \left| \hat{h}_q(m) - h_q(m) \right|, \left| \hat{h}_{q'}(m) - h_{q'}(m) \right| \right\},$$

the frequency in (7) does not exceed

$$\nu_n\Bigg(\sum_{\substack{p^\alpha \| m \\ \alpha \geqq 2}} \left| h(p^\alpha) \right| \geqq \varepsilon\beta(x) \Bigg) \leqq \nu_n\Bigg(\sum_{\substack{p^\alpha \| m \\ \alpha \geqq 2, p^\alpha > K}} \left| h(p^\alpha) \right| \geqq \varepsilon\beta(x) \Bigg) + o_K(1) =:$$
$$=: \nu_n + o_K(1)$$

as $n \to \infty$ and $x \to \infty$ for each $K > 2$. But

$$\nu_n \leqq \nu_n(m : \exists\, p^2 | m,\, p > K^{1/2}) \leqq \sum_{p \geqq K^{1/2}} p^{-2} << K^{-1/2}.$$

Since K is arbitrary, the above estimates imply the assertion of Lemma 2.

COROLLARY. *Proposition* (5) *for the function* $G_k(m,t)$ *is equivalent to that for the function* $\widehat{G}_k(m,t)$.

In what follows we shall use the function $\widehat{G}_k(m,t)$ omitting the "cap". Let $r \geqq 2$, $q_1 := \max\{q : q \leqq r\}$, and $H^r(m,t) = H(m,t)$ if $t \leqq D(q_1)$ and $H^r(m,t) = H\big(m, D(q_1)\big)$ if $t > D(q_1)$. Put $G_k^r(m,t) = \beta(k)^{-1} H^r(m, D(k)t)$ when $0 \leqq t \leqq 1$.

LEMMA 3. *There exists a sequence* $r = r(n)$ *such that* $r \to \infty$, $\log r = o(\log n)$ *as* $n \to \infty$ *and*

$$\lim_{x\to\infty} \limsup_{n\to\infty} \nu_n\left(\max_{x\leqq k\leqq n} \rho\big(G_k(m,\cdot), G_k^r(m,\cdot)\big) \geqq \varepsilon \right) = 0.$$

PROOF. We need to consider the difference $H(m,t) - H^r(m,t)$ in the interval $D(q_1) \leqq t \leqq D(k)$ only. Then in virtue of the definitions we have

$$\left| H(m,t) - H^r(m,t) \right| \leqq \max_{r \leqq l \leqq k} \left| h_l(m) - h_r(m) - \big(A(l) - A(r)\big) \right|.$$

Hence by Lemma 2 [11]

$$\nu_n\Big(\max_{x\leqq k\leqq n}\rho\big(G_k(m,\cdot),G_k^r(m,\cdot)\big)\geqq\varepsilon\Big)\leqq$$

$$\leqq\nu_n\Big(\max_{r\leqq l\leqq n}\big|h_l(m)-h_r(m)-\big(A(l)-A(r)\big)\big|\geqq\varepsilon\beta(r)\Big)<<\frac{D(n)-D(r)}{\beta^2(r)}.$$

Condition (4) implies that

$$\max_{p\leqq n}\big|h(p)\big|<<\sqrt{D(n)}\mu_n$$

with $\mu_n=\big(L_2D(n)\big)^{-1/2}\longrightarrow 0$ as $n\to\infty$. Thus, taking $r=n^{\mu_n}$ as in [9] we obtain that $D(n)-D(r)=o\big(D(r)\big)$ as $n\to\infty$. Lemma 3 is proved.

COROLLARY. *Let $r=r(n)$ be the sequence from Lemma 3. Then proposition (5) of Theorem 1 is equivalent to that with $G_k^r(m,t)$ in place of $G_k(m,t)$.*

Now we are in a position to apply the Kubilius model of the probabilistic space [9]. We do not need its construction in detail. Let ξ_p, where p runs over the sequence of primes, be a sequence of i. r. v.s such that

$$P\big(\xi_p=h(p)\big)=1-P\big(\xi_p=0\big)=1/p. \tag{8}$$

Denote by $\Xi(t)$ the process starting at the point $(0,0)$ and joining linearly the points $\big(D(q),\Sigma_q\big)$ where

$$\Sigma_u=\sum_{p\leqq u}\xi_p-A(u).$$

Let $\Xi^r(t)=\Xi(t)$ if $t\leqq D(q_1)$ and $\Xi^r(t)=\Xi\big(D(q_1)\big)$ if $t>D(q_1)$. We denote $\zeta_k(t)=\beta(k)^{-1}\Xi(D(k)t)$ and $\zeta_k^r(t)=\beta(k)^{-1}\Xi^r(D(k)t)$. Then we have the following partial case of the fundamental lemma [9].

LEMMA 4. *Let $r=r(n)$ be the sequence from Lemma 3. Then*

$$\nu_n\Big(\max_{x\leqq k\leqq n}\rho\big(G_k^r(m,\cdot),\mathcal{K}\big)\geqq\varepsilon\Big)=P\Big(\max_{x\leqq k\leqq n}\rho\big(\zeta_k^r(.),\mathcal{K}\big)\geqq\varepsilon\Big)+o(1) \tag{9}$$

and

$$(10)\quad \nu_n\Big(\min_{x\leqq k\leqq n}\rho\big(G_k^r(m,\cdot),g\big)<\varepsilon\Big)=P\Big(\min_{x\leqq k\leqq n}\rho\big(\zeta_k^r(.),g\big)<\varepsilon\Big)+o(1)$$

for each $\varepsilon>0$ *and* $g\in\mathcal{K}$ *as* $n\to\infty$.

PROOF. The events under the frequency notation ν_n in (9) and (10) belong to the algebra generated by the sets $\{m: m\equiv 0 \bmod p\}$, $p\leqq r$. For the remaining arguments see in [9].

LEMMA 5. *Let* $r=r(n)$ *be the sequence from Lemma* 3. *Then*

$$P\Big(\max_{x\leqq k\leqq n}\rho\big(\zeta_k(.),\zeta_k^r(.)\big)\geqq\varepsilon\Big)=o(1)$$

as $n\to\infty$ *and* $x\to\infty$.

The proof goes along the same lines as that of Lemma 3.

Now we need the following probabilistic result of P. Major [10]. For independent random variables X_i such that $EX_i=0$, $i=1,2,\ldots$, we put $S_n=X_1+\ldots+X_n$ and $B_n=EX_1^2+\ldots+EX_n^2$. Define the process $Y(t)$ starting at the point $(0,0)$ and joining linearly the points (B_n,S_n).

THEOREM A ([10]). *Let* $X_1,X_2,\ldots$ *be independent random variables given on a common probabilistic space,* $EX_i=0$, $i\geqq 1$. *Suppose that there exists a sequence of numbers* M_n *such that*

$$P(|X_n|\leqq M_n)=1$$

for each $n\geqq 1$, $M_n=o\big((B_n/L_2B_n)^{1/2}\big)$ *and* $B_n\to\infty$ *as* $n\to\infty$. *Then*

$$\frac{Y(B_nt)}{\sqrt{2B_nL_2B_n}}\Longrightarrow\mathcal{K}\qquad (P\text{-}a.s.).$$

PROOF OF THEOREM 1. In virtue of (4) the random variables ξ_p satisfy the conditions of Theorem A. Thus,

$$\zeta_k(t)\Longrightarrow\mathcal{K}\qquad (P\text{-a.s.}).$$

By Lemma 5 the same holds for $\zeta_k^r(t)$ in the place of $\zeta_k(t)$. Now the relations (9), (10), and Corollary of Lemma 3 imply the assertion of Theorem 1.

PROOF OF THEOREM 2. Let $\tau_{qk}=D(q)/D(k)$, $m\leqq n$, and $k\leqq n$. Then

$$\rho\big(U_k(m,\cdot),G_k(m,\cdot)\big)\leqq$$

$$\leqq \beta(k)^{-1} \max\Big\{ \big| h_q(m) - h_{q'}(m) - \big(A(q) - A(q')\big) \big| : \tau_{qk} \leqq 1 \Big\} \leqq$$
$$\leqq \beta(k)^{-1} \max\big\{ \big| h(q) \big| : q \leqq k \big\}.$$

Hence in virtue of condition (4) the assertion of Theorem 2 follows from that of Theorem 1.

To prove Theorem 3, at the first stage we can follow the way outlined in [7], pages 25 and 26. Instead of the sketch in the next lemma we present the necessary details.

LEMMA 6 ([7]). *Let $\tau(m,u,v) := \#\{d : d \big| m, d > u, p(d) \leqq v\}$ then uniformly in $2 \leqq u$, $v \leqq n$ we have*

$$\sum_{m \leqq n} \tau(m,u,v) << nLv \exp\big\{ -cLu/Lv \big\}, \qquad c > 0.$$

Further, for $\alpha = 1/L2$, $u_j := \exp\{j^\alpha\}$, $v_j := \exp\{cj^\alpha/3Lj\}$,

$$\nu_n\Big(\max\big\{ \tau(m,u_j,v_j) : \xi(x) \leqq j \leqq (Ln)^{1/\alpha} \big\} > 0 \Big) \longrightarrow 0$$

with $\xi(x) \to 0$ arbitrarily slowly.

PROOF. From the definition of $\tau(m,u,v)$ we have

$$\sum_{m \leqq n} \tau(m,u,v) \leqq n \sum_{\substack{d>u \\ p(d) \leqq v}} \frac{1}{d} \leqq nu^{-\lambda} \prod_{p \leqq v} \Big(1 - p^{\lambda - 1}\Big)^{-1} <<$$
$$<< n \exp\Big\{ -\lambda Lu + \sum_{p \leqq v} p^{\lambda - 1} \Big\}$$

provided that $0 < \lambda \leqq 1/3$. The choice $\lambda = c/Lv$ with sufficiently small $c > 0$ proves the first assertion of Lemma 6.

Since $\tau(m,u,v) \in \mathbf{N}$, the second proposition of Lemma 6 follows from the first one and the convergence of the series

$$\sum_j Lu_j \exp\big\{ -cLu_j/Lv_j \big\}.$$

Lemma 6 is proved.

PROOF OF THEOREM 3. Following the idea suggested in [7] we shall approximate $T_k(m,t)$ by processes considered in Theorem 2. Since $T_k(m,t) \leqq \leqq \Delta_k(m,t)$, $0 \leqq t \leqq 1$, and by Lemma 2

$$\rho\big(\Psi_k(m,\cdot),\Delta_k(m,\cdot)\big) \to 0 \qquad (\nu_n\text{-a.s.}),$$

we have the desired upper estimate. We shall express the lower estimate of $T_k(m,t)$ in terms of $\Psi_k(m,t)$ too. Let as previously $z = z_k(t) = \exp\left\{(Lk)^t\right\}$. Observe that for each $t_k > 0$ such that $z_k(t_k)$ remains bounded as $k \to \infty$ we trivially have

$$\nu_n\left(\max_{x\leqq k\leqq n}\sup_{0\leqq t\leqq t_k}\frac{\left|\log_2\tau\big(m,z_k(t)\big)-tL_2k\right|}{b(k)}\geqq\varepsilon\right)\longrightarrow 0.$$

On the other hand, by the Cauchy inequality and the definition of $\mathcal{K}$ we derive

$$\max\left\{\left|g(t)\right| : 0\leqq t\leqq t_x\right\}\leqq\left(t_x\int_0^{t_x}\left|g'(u)\right|du\right)^{1/2}\leqq t_x^{1/2}\longrightarrow 0$$

uniformly in $g \in \mathcal{K}$ as $t_x \to 0$. The last two estimates show that on the interval $[0,t_x]$ the sequence of arithmetical processes approaches the zero function belonging to $\mathcal{K}$. Therefore it remains to consider the convergence in the interval $[t_x,1]$ where $t_x \to 0$.

As it has been observed in [7], page 25,

$$2^{\omega(m,v)}\leqq\tau(m,u)+\tau(m,u,v),\qquad m,u,v\geqq 1.$$

We divide the interval $[z_x,n]$, in which we need to estimate $\tau(m,z)$ from below, by the points u_j, $j_x \leqq j \leqq J_n$, where $j_x \to \infty$ as $x \to \infty$. In virtue of the monotonicity of $\tau(m,z)$ with respect to z and Lemma 6, for $z \in (u_j,u_{j+1}]$, we obtain

$$\tau(m,z)\geqq\tau(m,u_j)\geqq 2^{\omega(m,v_j)}-\tau(m,u_j,v_j)=2^{\omega(m,v_j)}$$

uniformly in $j_x \leqq j \leqq J_n$ and ν_n-a.s. in m as $n,x \to \infty$. It remains to change v_j on the right-hand side by z.

Let as earlier $z = z_k(t) \in (u_j,u_{j+1}]$, then $b^2(k) \geqq \gamma^2(j) := 2\alpha(Lj)L_3j$ and

$$\omega(m,v_j)\geqq\omega(m,z)-\big(\omega(m,u_{j+1})-\omega(m,v_j)\big).$$

Later on we shall obtain the estimate

$$(11)\qquad \nu_n(\varepsilon):=\nu_n\left(\max_{j_x\leqq j\leqq J_n}\gamma(j)^{-1}\big(\omega(m,u_{j+1})-\omega(m,v_j)\big)>\varepsilon\right)=o(1)$$

for each $\varepsilon > 0$. Together with the last inequality this implies that $T_k(m,t) \geqq \Psi_k(m,t)$ uniformly in $t_x \leqq t \leqq 1$, $x \leqq k \leqq n$ and ν_n-a.s. in m. Recalling our remark about the interval $0 \leqq t \leqq t_x$ and the upper estimate of $T_k(m,t)$, we see that Theorem 3 follows from Corollary 1 of Theorem 2.

To prove (11), in contrast to the approach used in [7], we reduce this problem to that for independent r.v.'s. In virtue of $L\big((Lu_{j+1})/Lv_j\big) \sim L_2 j = o\big(\gamma(j)\big)$ as $j \to \infty$ applying Ruzsa's inequality (see [18], Theorem 2) we obtain

$$\nu_n(\varepsilon) << P\bigg(\max_{j_x \leqq j \leqq J_n} \gamma(j)^{-1}\bigg|\sum_{v_j < p \leqq u_{j+1}} \bar{\xi}_p\bigg| \geqq \varepsilon/4\bigg),$$

where ξ_p, $p \leqq n$, denote independent r.v.'s defined by (2) with $h(p) = 1$ and the bar indicates centering of r.v.'s by their expectations. Let for the sake of convenience $e(k) = \big[\exp\{k\}\big]$, $k_x = [Lj_x]$, $K_n = [LJ_n] + 1$,

$$X_s := \sum_{v_{e(k)+s-1} < p \leqq v_{e(k)+s}} \bar{\xi}_p, \qquad s = 1, \dots, e(k+1) - e(k) := S$$

and

$$Y_0 := \sum_{v_{e(k)} < p \leqq u_{e(k)}} \bar{\xi}_p,$$

$$Y_r := \sum_{u_{e(k)+r-1} < p \leqq v_{e(k)+r}} \bar{\xi}_p, \qquad r = 1, \dots, e(k+1) + 1 - e(k) =: R.$$

Then

$$\nu_n(\varepsilon) << \sum_{k_x \leqq k \leqq K_n} P\bigg(\max_{1 \leqq s \leqq S}\bigg|\sum_{l=1}^{s} X_l\bigg| \geqq \varepsilon\gamma\big(e(k)\big)/8\bigg) +$$

$$+ \sum_{k_x \leqq k \leqq K_n} P\bigg(\max_{0 \leqq r \leqq R}\bigg|\sum_{l=0}^{r} Y_l\bigg| \geqq \varepsilon\gamma\big(e(k)\big)/8\bigg).$$

Thus, we have an ordinary probabilistic problem. From the definitions of r.v.'s we observe that the variance of the sum of X_l, $1 \leqq l \leqq S$, is bounded uniformly in k, while the variance of the sum of Y_l, $0 \leqq l \leqq R$, increases as $Lk = o\big(\gamma\big(e(k)\big)\big)$ when $k \to \infty$. Therefore for sufficiently large k_x from the Lévy inequality (see [16], p. 68) we derive

$$\nu_n(\varepsilon) << \sum_{k_x \leqq k \leqq K_n} P\bigg(\bigg|\sum_{l=1}^{S} X_l\bigg| \geqq \varepsilon\gamma\big(e(k)\big)/9\bigg) +$$

$$+\sum_{k_x \leqq k \leqq K_n} P\Big(\Big|\sum_{l=0}^{R} Y_l\Big| \geqq \varepsilon\gamma\big(e(k)\big)/9\Big).$$

Now we apply the exponential estimates of the probabilities involved. Verifying the conditions of Theorem 15 of [16], p. 70, we have

$$\mathbf{E}\exp\{tX_l\} \leqq \exp\{\delta t^2 a_l\}$$

for $0 < t \leqq t_0$ with some positive t_0, $\delta = \delta(t_0)$, and a_l such that $a_1 + \ldots + a_S << 1$. Similarly,

$$\mathbf{E}\exp\{tY_l\} \leqq \exp\{\delta_1 t^2 b_l\},$$

where $\delta_1, b_l > 0$, $0 < t \leqq t_0$, and $b_0 + \ldots + b_R << Lk = o\big(\gamma(k)\big)$. If δ_2 is a sufficiently small positive constant, then by the aforementioned Theorem we obtain

$$\nu_n(\varepsilon) << \sum_{k_x \leqq k \leqq K_n} \exp\Big\{-\delta_2\gamma\big(e(k)\big)\Big\} \longrightarrow 0$$

as n and x tend to infinity. This is the desired estimate (11).

Theorem 3 is proved.

3. Inverses

Let $\mathbf{D}_0$ denote the subspace of $\mathbf{D} = \mathbf{D}[0,1]$ consisting of nonnegative nondecreasing functions. For $X \in \mathbf{D}_0$, we define $X_1^{-1}, X_2^{-1} \in \mathbf{D}_0$ by

$$X_1^{-1}(t) = \sup\{u \in [0,1] : X(u) \leqq t\},$$

$$X_2^{-1}(t) = \inf\{u \in [0,1] : X(u) > t\}.$$

Here and in what follows we put $X_2^{-1}(t) = 1$ for $X(1) \leqq t \leqq 1$.

LEMMA 7. *Let $X_n \in \mathbf{D}_0$ and δ_n be a sequence of positive numbers, $\delta_n \to \to 0$. If $Y \in \mathbf{C}[0,1]$, then as $n \to \infty$ in both cases $j = 1$ or 2 the following two relations are equivalent:*

$$\sup\left\{\left|\frac{X_n(t) - t}{\delta_n} - Y(t)\right| : t \in [0,1]\right\} \longrightarrow 0$$

and

$$\sup\left\{\left|\frac{X_{nj}^{-1}(t) - t}{\delta_n} + Y(t)\right| : t \in [0,1]\right\} \longrightarrow 0.$$

The proof of Lemma 7 is the same as that of Theorem 3.1.4 in [21].

PROOF OF THEOREM 4. We apply Lemma 6 for the function

$$X_k(m,t) = \omega\left(m, \exp\left\{(Lk)^t\right\}\right)/L_2k$$

and $\delta_k = \left((2L_4k)/L_2k\right)^{1/2}$. Observe that, for $k_0 \leqq k < \exp\{\exp\{\omega(m)\}\}$ and $0 \leqq t \leqq 1$,

$$X_{k1}^{-1}(m,t) = (L_2k)^{-1}L\big(\sup\{1 \leqq z \leqq Lk : \omega(m,e^z) \leqq tL_2k\}\big) =$$
$$= (L_2k)^{-1}L_2\big(\sup\{e \leqq w \leqq k : \omega(m,w) \leqq tL_2k\}\big) \leqq$$
$$\leqq (L_2k)^{-1}L_2\big(\sup\{w \geqq 0 : \omega(m,w) \leqq tL_2k\}\big) =$$
$$= (L_2k)^{-1}L_2\big(p(m,tL_2k+1)-1\big).$$

Similarly,

$$X_{k2}^{-1}(m,t) = (L_2k)^{-1}L_2\big(\inf\{e \leqq w \leqq k : \omega(m,w) > tL_2k\}\big) \geqq$$
$$\geqq (L_2k)^{-1}L_2\big(\inf\{w \geqq 0 : \omega(m,w) > tL_2k\}\big) = (L_2k)^{-1}L_2p(m,tL_2k+1).$$

Since according to Lemma 7 for almost all m the cluster set of the sequence of functions $\left(X_{kj}^{-1}(m,t)-t\right)/\delta_k$ agrees with $\mathcal{K}$, the same holds for $\left((L_2k)^{-1}L_2p(m,tL_2k+1)-t\right)/\delta_k$. This, in virtue of $L_2p(m,j+1) \sim L_2p(m,j)$ for almost all m as $j \to \infty$, implies Theorem 4.

PROOF OF THEOREM 5. If now

$$X_k(m,t) = \left(\log_2 \tau\left(m, \exp\left\{(Lk)^t\right\}\right)\right)/L_2k,$$

then for its generalized inverses introduced in Lemma 7 we have

$$X_{k1}^{-1}(m,t) \leqq (L_2k)^{-1}L_2\big(d(m,2^{tL_2k}+1)-1\big)$$

and

$$X_{k2}^{-1}(m,t) \geqq (L_2k)^{-1}L_2d(m,2^{tL_2k}+1).$$

Thus, by Lemma 7 the desired result follows from Theorem 4.

Addendum. To the Exercises in Chapter 1 of [7] we would suggest to add the following relations:

(1) uniformly in $x \leqq j \leqq n$, ν_n-a.s in m, and for each $s \geqq 5$, we have

$$\left|L_2d(m,j) - \frac{Lj}{L2}\right| \leqq \left(\frac{2Lj}{L2}\left(L_3j + \frac{3}{2}L_4j + L_5j + \ldots + (1+\varepsilon)L_sj\right)\right)^{1/2}$$

when n and x tend to infinity;

(2) we have

$$\lim_{x\to\infty}\limsup_{n\to\infty}\nu_n\left(\min_{x\leqq j\leqq\omega(m)}\frac{\sqrt{8L_2j}}{\pi\sqrt{j}}\max_{l\leqq j}\left|L_2p(m,l)-l\right|\leqq 1-\varepsilon\right)=0$$

but

$$\lim_{x\to\infty}\liminf_{n\to\infty}\nu_n\left(\min_{x\leqq j\leqq\omega(m)}\frac{\sqrt{8L_2j}}{\pi\sqrt{j}}\max_{l\leqq j}\left|L_2p(m,l)-l\right|\leqq 1+\varepsilon\right)=1;$$

(3) we have

$$\lim_{x\to\infty}\limsup_{n\to\infty}\nu_n\left(\min_{x\leqq j\leqq\tau(m)}\frac{\sqrt{8(L2)L_3j}}{\pi\sqrt{Lj}}\max_{l\leqq j}\left|L_2d(m,l)-\frac{Ll}{L2}\right|\leqq 1-\varepsilon\right)=0$$

but

$$\lim_{x\to\infty}\liminf_{n\to\infty}\nu_n\left(\min_{x\leqq j\leqq\tau(m)}\frac{\sqrt{8(L2)L_3j}}{\pi\sqrt{Lj}}\max_{l\leqq j}\left|L_2d(m,l)-\frac{Ll}{L2}\right|\leqq 1+\varepsilon\right)=1.$$

Finally, we frankly confess that our attempt to extend Exercises 2 and 3 into a functional form have failed.

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DEPARTMENT OF PROBABILITY AND NUMBER THEORY
VILNIUS UNIVERSITY
NAUGARDUKO STR. 24
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LITHUANIA