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A FUNCTIONAL LIMIT THEOREM RELATED TO NATURAL DIVISORS

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Functional limit distributions related to additive functions, which by the very definition are determined by their values on prime divisors of natural numbers, were intensively studied (see, for instance, [4, 11, 12] and references therein). In [6, 7] these investigations were applied to random processes defined in terms of natural divisors. As it was mentioned in a lecture in November, 1993, at the University of Bordeaux I (see also [5], [7]) of the first author, recent investigations of natural divisors done by G. Tenenbaum and others [2, 8, 9, 10] can be also treated from the point of view of probabilistic functional limit theorems.

Let $\mathbf{D}[0,1]$ be the space of real-valued functions on $[0,1]$ which are right-continuous and have left-hand limits. Suppose that the Skorokhod topology is introduced in $\mathbf{D}[0,1]$ and $\mathcal{D}$ is the Borel σ-algebra (see [1]). For a nonnegative multiplicative function $f(d)$, we put

$$F(m,v)=\sum_{d|m,\, d\leqq v} f(d), \quad F(m,m)=F(m),$$

where $m,d\in\mathbf{N}$. If $0\leqq t\leqq 1$, then

$$X_n:=X_n(m,t):=F(m)^{-1}F(m,n^t)\in\mathbf{D}[0,1].$$

Further, if $\nu_n(\ldots)$ denotes the frequency of $m\leqq n$ satisfying the condition written instead of the dots, then the distributions $\nu_n(X_n\in B)$, where $B\in\mathcal{D}$, define a probability measure on $\mathcal{D}$. We denote it by $\nu_n\cdot X_n^{-1}$. Now one can ask: *Under what conditions the sequence $\nu_n\cdot X_n^{-1}$ converges weakly to a limit measure when $n\to\infty$?*

Before giving an answer, we quote two well-known results. Let $\tau(m,v)$ be the number of natural divisors of $m\in\mathbf{N}$ which do not exceed $v\geqq 1$ and $\tau(m) = \tau(m,m)$. The theorem of J.-M. Deshouillers, F. Dress, and G. Tenenbaum

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[2] asserts that

$$\frac{1}{n}\sum_{m=1}^{n}\tau(m)^{-1}\tau(m,n^t)=\frac{2}{\pi}\arcsin\sqrt{t}+o(1)$$

uniformly in $t\in[0,1]$ as $n\to\infty$. Moreover [9], the sequence of distribution functions

$$\nu_n\Big(\tau(m)^{-1}\big(\tau(m,n^t)-\tau(m,n^s)\big)<x\Big)$$

weakly converges to a purely discrete limiting distribution as $n\to\infty$ for all $0\leqq s<t\leqq 1$. Evidently, the first relation deals with the expectation while the second one considers the increments of the processes defined above when $f(d)\equiv 1$. That gives also some motivation to our investigation.

In the present paper we will prove the following result.

THEOREM. *If $f(p)=\varkappa>0$ and $f(p^k)\geqq 0$ for all prime numbers p and $k\geqq 2$, then $\nu_n\cdot X_n^{-1}$ weakly converges to a limit measure defined on $\mathcal{D}$.*

PROOF. According to the criteria ([1], Theorem 15.4) we must verify that

(I) $G_n(T):=G_n(T;x_1,\dots,x_k):=\nu_n\big(X_n(m,t_1)<x_1,\dots,X_n(m,t_k)<x_k\big)$ *weakly converges as $n\to\infty$ to a k-dimensional distribution function for each $k\geqq 1$ and any fixed $T:=(t_1,\dots,t_k)\in[0,1]^k$, $t_1<\dots<t_k$ except some rare set,* and

(II) *the sequence of measures $\nu_n\cdot X_n^{-1}$ is tight.*

In what follows we will use a few well known estimates.

LEMMA 1 ([3]). *Let $g(m)$ be a multiplicative function, $0\leqq g(p^k)\leqq A$ for all prime numbers p and $k\geqq 1$. Then for $n\geqq 2$,*

$$\sum_{m\leqq n}g(m)\ll_A\frac{n}{\log n}\exp\Big\{\sum_{p\leqq n}\frac{g(p)}{p}\Big\}\tag{1}$$

and

$$\sum_{m\leqq n}\frac{g(m)}{m}\ll_A\exp\Big\{\sum_{p\leqq n}\frac{g(p)}{p}\Big\}.\tag{2}$$

As we will see, the contribution of numbers m having small prime factors to the distribution of X can be neglected. For $0<\varepsilon<1$, we denote

$$m(\varepsilon)=\prod_{\substack{p^\alpha\| m\\ p>n^\varepsilon}}p^\alpha$$

and $\vartheta = \log 1/\varepsilon$. For brevity write $a = m/m(\varepsilon)$, $b = m(\varepsilon)$, and let $\rho(v)$ stand for the Dickman–de Bruijn function. In what follows summation over $a \leqq y$ (respectively $b \leqq y$) will denote that $a = a(m)$ ($b = b(m)$) runs over all possible values of the numbers not exceeding y and having small (large) prime factors. We have

LEMMA 2. *The estimates*

$$\sum_{a\leqq y} 1 - y\rho\left(\frac{\log y}{\varepsilon \log n}\right) \ll \frac{y}{\varepsilon \log n}, \qquad \sum_{b \leqq y} 1 \ll \frac{y}{\varepsilon \log n}, \qquad \sum_{b \leqq n} \frac{1}{b} \ll \varepsilon^{-1}$$

hold uniformly in $n^{\varepsilon} \leqq y \leqq n$.

PROOF. See, for instance [9], pages 8 and 10.

LEMMA 3. *Let* $\omega(m)$ *be the number of different prime factors of* m, $\vartheta = \log 1/\varepsilon$. *If* $c > 1$, *then*

$$\nu_n\big(m : \omega\big(m(\varepsilon)\big) > c\vartheta\big) \ll_c \vartheta^{-1}.$$

If $c_1 < 1$, $s > 1/(1 - c_1)$, *and* $n^{\sqrt[s]{\varepsilon}} \leqq y \leqq 2n$, *then*

$$\nu_y\big(m : \omega\big(m(\varepsilon)\big) < c_1\vartheta\big) \ll_{s,c_1} \vartheta^{-1}.$$

PROOF. The propositions follow from the Turán–Kubilius inequality

$$\sum_{m \leqq y}\left(\omega\big(m(\varepsilon)\big) - \sum_{n^{\varepsilon} \leqq p \leqq y} \frac{1}{p}\right)^2 \ll y \sum_{n^{\varepsilon} \leqq p \leqq y} \frac{1}{p}$$

and the observation that the sum over primes does not exceed $\vartheta + o(1)$ and is not less than $(1 - 1/s)\vartheta + o(1)$ when y belongs to the given interval.

LEMMA 4. *We have*

$$\nu_n\big(X_n(m,t) \neq X_n\big(m(\varepsilon), t\big)\big) \ll \vartheta^{-1/2}$$

uniformly in t, $\sqrt[s]{\varepsilon} \leqq t \leqq 1 - \sqrt[s]{\varepsilon}$ *for* $n \geqq n_0(\varepsilon)$.

PROOF. Via the equality

$$X_n(m,t) = \frac{1}{F(m)} \sum_{d_1 \mid \frac{m}{m(\varepsilon)}} f(d_1) \sum_{\substack{d_2 \mid m(\varepsilon) \\ d_2 \leqq n^t}} f(d_2)$$

$$- \frac{1}{F(m)} \sum_{d_1 \mid \frac{m}{m(\varepsilon)}} f(d_1) \sum_{\substack{d_2 \mid m(\varepsilon) \\ n^t/d_1 < d_2 \leqq n^t}} f(d_2),$$

we obtain

$$(3)\qquad \nu_n\big(X_n(m,t) \neq X_n(m(\varepsilon),t)\big) \leqq \nu_n\left(m : m > n^{\varepsilon\sqrt{\vartheta}} m(\varepsilon)\right)$$

$$+\nu_n\left(m : m \leqq n^{\varepsilon\sqrt{\vartheta}} m(\varepsilon), \exists\, d_2 | m(\varepsilon), \frac{m(\varepsilon)n^t}{m} \leqq d_2 \leqq n^t\right) =: \nu' + \nu''.$$

We have

$$\nu' \leqq \frac{1}{n\varepsilon\sqrt{\vartheta}\log n} \sum_{m\leqq n} \log\frac{m}{m(\varepsilon)} \leqq \frac{1}{\varepsilon\sqrt{\vartheta}\log n} \sum_{r\geqq 1, p\leqq n^{\varepsilon}} \frac{\log p^r}{p^r} \ll \frac{1}{\sqrt{\vartheta}}.$$

Now we will show that the same estimate holds for the second summand in (3) as well. If $m(\varepsilon) =: d_2 b_2$, where

$$m(\varepsilon)n^t/m \leqq d_2 \leqq n^t, \quad m \leqq n^{\varepsilon\sqrt{\vartheta}} m(\varepsilon),$$

we can suppose that

$$n^{1-t-2\varepsilon\sqrt{\vartheta}} \leqq b_2 \leqq n^{1-t}.$$

Otherwise we had $m \leqq n^{1-\varepsilon\sqrt{\vartheta}}$, and hence $\nu'' \leqq n^{-\varepsilon\sqrt{\vartheta}}$.

By Lemma 3 for all but $O(n\vartheta^{-1})$ numbers $m \leqq n$ we have $\omega(m(\varepsilon)) \leqq 3\vartheta/2$. Hence either $\omega(d_2) \leqq 3\vartheta/4$ or $\omega(b_2) \leqq 3\vartheta/4$, and these m can be expressed as $m = db(\varepsilon)$, where all the prime factors of $b(\varepsilon)$ belong to the interval $[n^{\varepsilon}, n]$ and

$$d \in D_t =: \left\{d : \omega(d(\varepsilon)) \leqq \frac{3}{4}\vartheta,\ n^{t-2\varepsilon\sqrt{\vartheta}} \leqq d \leqq n^{t+\varepsilon\sqrt{\vartheta}}\right\}$$

with some $t \in \left[\sqrt[6]{\varepsilon}, 1 - \sqrt[6]{\varepsilon}\right]$. Hence

$$\nu_n'' \ll \frac{1}{n} \max_{\sqrt[6]{\varepsilon}\leqq t\leqq 1-\sqrt[6]{\varepsilon}} \sum_{d\in D_t} \sum_{b(\varepsilon)\leqq n/d} 1 + \vartheta^{-1} \ll \frac{1}{\varepsilon\log n} \max_{\sqrt[6]{\varepsilon}\leqq t\leqq 1-\sqrt[6]{\varepsilon}} \sum_{d\in D_t} \frac{1}{d} + \vartheta^{-1}$$

by Lemma 2. Splitting the sum into the parts over the intervals $[Q, 2Q]$ with $Q = 2^j n^{t-2\varepsilon\sqrt{\vartheta}}$ and $j = 0, 1, \ldots, J$, where $J \ll \varepsilon\sqrt{\vartheta}\log n$, we obtain

$$\nu_n'' \ll \sqrt{\vartheta} \max_{n^{\sqrt[5]{\varepsilon}}\leqq Q\leqq n} \nu_{2Q}\big(d : \omega(d(\varepsilon)) \leqq 3\vartheta/4\big) + \vartheta^{-1}.$$

Now the desired estimate follows from Lemma 3. Lemma 4 is proved.

We will also use Lemma 9 of [8].

LEMMA 5. *Let $h:\mathbf{R}^l \to \mathbf{R}$ be a Riemann integrable function with support contained in a compact subset of $(0,\infty)^l$. Then*

$$\lim_{n\to\infty} \sum_{p_1,\dots,p_l} \frac{1}{p_1\cdots p_l} h\left(\frac{\log p_1}{\log n},\dots,\frac{\log p_l}{\log n}\right) = \int_{\mathbf{R}^l} h(u_1,\dots,u_l)\frac{du_1}{u_1}\cdots\frac{du_l}{u_l}.$$

Now we return to the proof of the Theorem. At first we will prove Proposition (I) for each $T=(t_1,\dots,t_k)$. Observe that without loss of generality we can assume $t_k<1$. To settle the case $t_1=0$, we evaluate the difference

$$0 \leqq \nu_n\big(X_n(m,t_2)<x_2,\dots,X_n(m,t_k)<x_k\big)$$
$$-\nu_n\big(X_n(m,0)<x_1, X_n(m,t_2)<x_2,\dots,X_n(m,t_k)<x_k\big)$$
$$\leqq \nu_n\big(X_n(m,0)\geqq x_1\big) \leqq \frac{1}{nx_1}\sum_{m=1}^{n}\frac{1}{F(m)},$$

where x_1 is an arbitrary positive number. Since $F(p)=1+\varkappa$ for each prime number p, in virtue of estimate (1) of Lemma 1 the last mean-value is of order $O\big((\log n)^{-\varkappa/(1+\varkappa)}\big)$. Thus, in what follows we take $t_1>0$. For any fixed ε, $0<\varepsilon<\min\left\{t_1^6,(1-t_k)^6\right\}$, from Lemma 4 we have

$$G_n(T)=\nu_n\big(X_n(m(\varepsilon),t_1)<x_1,\dots,X_n\big(m(\varepsilon),t_k\big)<x_k\big)+O(\vartheta^{-1/2})+o(1)$$
$$=:\nu_n\big(\Delta_n\big(m(\varepsilon),T\big)\in B\big)+O(\vartheta^{-1/2})+o(1)$$

as $n\to\infty$. Further, using the decomposition $m=ab$, where the prime divisors of a do not exceed n^ε and those of b are greater than n^ε, in virtue of Lemma 2 we obtain

$$G_n(T)=\frac{1}{n}\sum_{\substack{b\leqq n\\ \Delta_n(b,T)\in B}}\ \sum_{a\leqq n/b} 1+O(\vartheta^{-1/2})+o(1)$$
$$=\sum_{\substack{b\leqq n\\ \Delta_n(b,T)\in B}} b^{-1}\rho\left(\frac{\log(nb^{-1})}{\varepsilon\log n}\right)+O(\vartheta^{-1/2})+o(1).$$

The summands of the main term with respect to b having a squared prime divisor p can be estimated by

$$O\Big(\sum_{p>n^\varepsilon}\frac{1}{p^2}\Big)=o(1).$$

Hence

$$G_n(T) = \sum_{l \leqq \varepsilon^{-1}} S_l + O(\vartheta^{-1/2}) + o(1),$$

where

$$S_l := \sum_{\substack{b \leqq n \\ \Delta_n(b,T) \in B}}^{*} b^{-1} \rho\left(\frac{\log(nb^{-1})}{\varepsilon \log n}\right)$$

and * stands for the condition "b consists of exactly l different prime numbers". Let $\delta_i \in \{0,1\}$, $1 \leqq i \leqq l$, and denote $I_j(L)$ the indicator function of the event $\{L \leqq t_j\}$, $1 \leqq j \leqq k$. Then the conditions in the definition of S_l define the region $D_l(\varepsilon)$ for the vector $(\log p_1, \cdots, \log p_l)1/\log n$ contained in $[\varepsilon, 1]^l$. More precisely,

$$D_l(\varepsilon) = \bigcap_{1 \leqq j \leqq k} D_{lj} \bigcap \{(u_1, \ldots, u_l) : u_1 + \ldots + u_l \leqq 1, \varepsilon \leqq u_i \leqq 1\}$$

and

$$D_{lj} = \left\{(u_1, \ldots, u_l) : \sum_{\delta_1, \ldots, \delta_l} \varkappa^{\delta_1 + \cdots + \delta_l} I_j(\delta_1 u_1 + \cdots + \delta_l u_l) < x_j(1+\varkappa)^l\right\}.$$

By Lemma 5 we have

$$S_l = \int_{D_l(\varepsilon)} \rho\left(\frac{1}{\varepsilon}\left(1 - \sum_{i=1}^{l} u_i\right)\right) \frac{du_1}{u_1} \cdots \frac{du_l}{u_l} + o(1).$$

Hence we have the relation

$$G_n(T; x_1, \ldots, x_k) = W_\varepsilon(T; x_1, \ldots, x_k) + O(\vartheta^{-1/2}) + o(1)$$

uniformly in $x_1, \ldots, x_k$ as $n \to \infty$ for each sufficiently small ε.

Taking at first $n \to \infty$ and later $\varepsilon \to 0$ in virtue of the notation $\vartheta = \log 1/\varepsilon$ one can verify that the limit of G_n exists and coincides with

$$\lim_{\varepsilon \to 0} W_\varepsilon(T; x_1, \ldots, x_k)$$

which is a k-dimensional distribution function for each vector T. The assertion (I) is proved.

In order to prove (II), we use the criteria given in Theorem 15.3 [1]. Preserving the notations of the book [1] we consider

$$w''_X(m,\delta) = \sup \min \left\{ \left| X_n(m,t) - X_n(m,t_1) \right|, \left| X_n(m,t_2) - X_n(m,t) \right| \right\},$$

where the supremum is taken over all t_1, t, and t_2 such that $t_1 \leqq t \leqq t_2$, $t_2 - t_1 \leqq \delta$. Observe that

$$w''_X(m,\delta) = \sup \Big\{ \min \left\{ \left| X_n(m,t) - X_n(m,t-\delta) \right|, \right.$$

$$\left. \left| X_n(m,t+\delta) - X_n(m,t) \right| \right\};\ \delta \leqq t \leqq 1-\delta \Big\},$$

$$w_X\big(m,\ [a,b)\big) := \sup \Big\{ \left| X_n(m,t) - X_n(m,t_1) \right|;\ a \leqq t_1 \leqq t \leqq b \Big\}$$

$$= \left| X_n(m,b) - X_n(m,a) \right|.$$

Thus, the criteria for the assertion (II) reduces to

(III) *For each positive* ε *and* η *there exists* δ, $0 < \delta < 1/2$, *such that*

$$\nu_n\big(X_n(m,\delta) \geqq \varepsilon\big) \leqq \eta, \tag{4}$$

$$\nu_n\big(X_n(m,1) - X_n(m,1-\delta) \geqq \varepsilon\big) \leqq \eta, \tag{5}$$

and

$$\nu_n\big(w''_X(m,\delta) \geqq \varepsilon\big) \leqq \eta \tag{6}$$

provided that n *is sufficiently large.*

All our calculations will be based on moment estimates. We have

$$\nu_n\big(X_n(m,\delta) \geqq \varepsilon\big) \leqq \frac{1}{n\varepsilon} \sum_{m=1}^{n} \frac{1}{F(m)} \sum_{\substack{d|m \\ d \leqq n^\delta}} f(d) = \frac{1}{n\varepsilon} \sum_{d \leqq n^\delta} \frac{f(d)}{F(d)} \sum_{k \leqq n/d} \frac{F(d)}{F(kd)}.$$

But the multiplicative function $g(m) = F(d)/F(md)$ satisfies the conditions $g(p) = 1/(1+\varkappa)$ if $p \nmid d$ and $g(p) \leqq 1$ if $p|d$. Hence estimate (1) of Lemma 1 implies

$$\nu_n\big(X_n(m,\delta) \geqq \varepsilon\big) \ll \varepsilon^{-1} (\log n)^{-\varkappa/(1+\varkappa)} \sum_{d \leqq n^\delta} \frac{f(d)e(d)}{dF(d)}.$$

Here and in what follows

$$e(d) := \exp\Big\{ \sum_{p|d} \frac{1}{p} \Big\}.$$

Now inequality (2) of Lemma 1 implies

$$\nu_n\big(X_n(m,\delta) \geqq \varepsilon\big) \ll \varepsilon^{-1}\delta^{\varkappa/(1+\varkappa)}. \tag{7}$$

(4) is proved.

Similarly,

$$\nu_n\big(X_n(m,1) - X_n(m,1-\delta) \geqq \varepsilon\big) \leqq \frac{1}{n\varepsilon} \sum_{n^{1-\delta}<d\leqq n} \frac{f(d)}{F(d)} \sum_{k\leqq n/d} \frac{F(d)}{F(kd)}$$

$$\ll \frac{1}{\varepsilon} \sum_{n^{1-\delta}<d\leqq n} \frac{f(d)}{dF(d)} \left(\log\frac{2n}{d}\right)^{-\varkappa/(1+\varkappa)}.$$

By Abel summation the last sum is equal to

$$-\sum_{n^{1-\delta}<d\leqq n} \frac{f(d)}{F(d)} \int_d^n d\left(\frac{1}{u}\left(\log\frac{2n}{u}\right)^{-\varkappa/(1+\varkappa)}\right)$$

$$+\frac{(\log 2)^{-\varkappa/(1+\varkappa)}}{n} \sum_{n^{1-\delta}<d\leqq n} \frac{f(d)}{F(d)}$$

$$\ll \int_{n^{1-\delta}}^n \sum_{n\leqq u} \frac{f(d)}{F(d)} \left(\log\frac{2n}{u}\right)^{-\varkappa/(1+\varkappa)} \frac{du}{u^2} + O\big((\log n)^{-1/(1+\varkappa)}\big).$$

Hence in virtue of Lemma 1 we obtain

$$\nu_n\big(X_n(m,1) - X_n(m,1-\delta) \geqq \varepsilon\big) \tag{8}$$

$$\leqq \frac{1}{\varepsilon\log n} \int_{n^{1-\delta}}^n \left(1 - \frac{\log u}{\log n}\right)^{-\varkappa/(1+\varkappa)} \frac{du}{u}$$

$$= \frac{1}{\varepsilon} \int_{1-\delta}^1 (1-v)^{-\varkappa/(1+\varkappa)} dv \ll \varepsilon^{-1}\delta^{1/(1+\varkappa)}.$$

(5) is proved.

To prove (6), we split the domain for t into several parts as follows. Let $\delta_1 = (\log 1/\delta)^{-1}$. We obtain from (7) and (8)

$$\nu_n\Big(\sup_{t\leqq\delta_1}\big(X_n(t+\delta)-X_n(t)\big)\geqq\varepsilon\Big) \tag{9}$$

$$+\nu_n\Big(\sup_{1-\delta_1\leqq t\leqq 1}\big(X_n(t)-X_n(t-\delta)\big)\geqq\varepsilon\Big)\ll\varepsilon^{-1}\big(\delta_1^{\varkappa/(1+\varkappa)}+\delta_1^{1/(1+\varkappa)}\big).$$

Further,

$$\nu''':=\nu_n\Big(\sup_{1/2-\delta_1\leqq t\leqq 1/2+\delta_1}\big(X_n(t+\delta)-X_n(t)\big)\geqq\varepsilon\Big)$$

$$\leqq\frac{1}{\varepsilon n}\sum_{m=1}^{n}\frac{1}{F(m)}\sum_{\substack{d|m\\ n^{1/2-\delta_1}\leqq d\leqq n^{1/2+\delta_1}}}f(d)$$

$$=\frac{1}{\varepsilon n}\sum_{n^{1/2-\delta_1}\leqq d\leqq n^{1/2+\delta_1}}\frac{f(d)}{F(d)}\sum_{m\leqq n/d}\frac{F(d)}{F(dm)}.$$

By Lemma 1,

$$\nu'''\ll\frac{(\log n)^{\varkappa/(1+\varkappa)}}{\varepsilon}\sum_{n^{1/2-\delta_1}\leqq d\leqq n^{1/2+\delta_1}}\frac{f(d)}{dF(d)}.$$

As in the estimation of ν'', we split the interval $[n^{1/2-\delta_1}, n^{1/2+\delta_1}]$ into $O(\delta_1\log n)$ subintervals $[M,2M]$. Then using (1) we obtain

$$\nu'''\ll\varepsilon^{-1}(\log n)^{-\varkappa/(1+\varkappa)}\delta_1\log n\max_{n^{1/4}\leqq M\leqq n}\frac{1}{M}\sum_{d\leqq 2M}\frac{f(d)}{F(d)}\ll\frac{\delta_1}{\varepsilon}. \tag{10}$$

According to (9) and (10) it remains to take the supremum in $w''_X(m,\delta)$ over $t\in[\delta_1,1/2-\delta_1]\cup[1/2+\delta_1,1-\delta_1]=:I(\delta_1)$. In virtue of Lemma 4 in this case up to the remainder term $O\big(\sqrt{\delta_1}\big)$ we can replace $w_X(m,\delta))$ by $w_X\big(m(\delta),\delta\big)\big)$. Here as previously, $m(\delta)$ stands for the product of prime factors of m which are greater than n^δ. Moreover, we can suppose that all the primes $p>n^\delta$ divide m at most in the first power,

$$n/\log n\leqq m\leqq m(\delta)n^{\delta\sqrt{\vartheta}},\quad m(\delta)\geqq n^{1-2\delta\sqrt{\vartheta}}, \tag{11}$$

where now $\vartheta = \log 1/\delta = \delta_1^{-1}$. Thus, if we define a new multiplicative function $y(m)$ by $y(p^r) = 0$ when $p \leqq n^\delta$ or $r \geqq 2$ and $y(p) = f(p) = \varkappa$ when $p > n^\delta$, and put

$$Y_n(m,t) = \frac{1}{\Phi(m)} \sum_{d|m, d \leqq n^t} y(d), \quad \Phi(m) = \sum_{d|m} y(d),$$

then

$$\nu_n\big(w''_X(m,\delta) \geqq \varepsilon\big) = \nu_n\big(w''_Y(m,\delta) \geqq \varepsilon\big) + O\big(\sqrt{\delta_1}\big)$$

provided $n \geqq n_0(\delta)$.

Since $\sup_t \min\{u(t), v(t)\} \geqq 1$ implies $\sup_t \min\{u(t), v(t)\} \leqq \sup_t \big(u(t)v(t)\big)$ for each pair of functions $u(t)$ and $v(t)$, we have

$$\nu_n\big(w''_Y(m,\delta) \geqq \varepsilon\big)$$
$$\leqq \frac{1}{\varepsilon^2 n} \sideset{}{^*}\sum_{m \leqq n} \sup_{t \in I(\delta_1)} \big(Y_n(m,t) - Y_n(m,t-\delta)\big)\big(Y_n(m,t+\delta) - Y_n(m,t)\big).$$

Here * over the sum denotes that m satisfies the condition (11) and $m(\delta)$ is squarefree. The quantity under the supremum is the sum of $y(d_1)y(d_2)\Phi^{-2}(m)$ over the set

$$D(m) := \{(d_1, d_2) : \ d_1, d_2 | m, \ n^{t-\delta} \leqq d_1 \leqq n^t, \ n^t \leqq d_2 \leqq n^{t+\delta}\}.$$

Now we use the representations $m = ab$, $d_1 = a_1 b_1$, $d_2 = a_2 b_2$, where none of the prime factors of a, a_1, a_2 exceed n^δ and those of b, b_1, b_2 are greater than n^δ. In virtue of condition (11) for $(d_1, d_2) \in D(m)$, we have $b_1 \leqq a_2 b_2 \leqq b_2 n^{\delta\sqrt{\vartheta}}$. If $t \in I(\delta_1)$ and (11) holds, then

$$b_1 \geqq (a_2 b_2)^{(t-\delta)/t+\delta)} a_1^{-1} \geqq b_2 b_2^{-2\delta/(t+\delta)} a_1^{-1} \geqq b_2 n^{-\delta_3}, \quad \delta_3 := 3\delta \log 1/\delta$$

provided δ is sufficiently small. Together with the previous estimate we get

$$b_1 \in B(b_2) := [b_2 n^{-\delta_3}, b_2 n^{\delta_3}]. \tag{12}$$

To simplify the conditions for b_2 we observe that $n^t \leqq a_2 b_2 \leqq n^{t+\delta}$ implies

$$b_2 \in B := \big[n^{\delta_1/2}, n^{(1-\delta_1)/2}\big] \tag{13}$$

when $\delta_1 \leqq t \leqq 1/2 - \delta_1$.

When $1/2+\delta_1 \leqq t \leqq 1-\delta_1$, we will substitute d_1 and d_2 by m/d_2 and m/d_1, respectively. Then (11) and the inequalities

$$n^{t-\delta} \leqq \frac{m}{a_2 b_2} \leqq n^t, \quad n^t \leqq \frac{m}{a_1 b_1} \leqq n^{t+\delta}, \quad 1/2+\delta_1 \leqq t \leqq 1-\delta_1$$

imply (12) and (13) as well.

Now

$$\nu_n\big(w_Y''(m,\delta) \geqq \varepsilon\big)$$

$$\leqq \frac{1}{\varepsilon^2 n} \sum_{ab \leqq n} \frac{1}{\Phi^2(a)\Phi^2(b)} \sum_{a_1,a_2|a} \Bigg(y(a_1)y(a_2) \sum_{\substack{b_2|b \\ b_2\in B}} f(b_2) \sum_{\substack{b_1|b \\ b_1\in B(b_2)}} f(b_1)$$

$$+y\left(\frac{a}{a_2}\right) y\left(\frac{a}{a_1}\right) \sum_{\substack{b_2|b \\ b_2\in B}} f\left(\frac{b}{b_2}\right) \sum_{\substack{b_1|b \\ b_1\in B(b_2)}} f\left(\frac{b}{b_1}\right)\Bigg).$$

Here as in the sum with * above, we remain assuming that $m(\delta)$ is squarefree. According to the definition of the function y, we have $y(a_i)=y(a/a_i)=0$, $i=1,2$, except the case $y(1)=1$. Hence

$$\nu_n\big(w_Y''(m,\delta) \geqq \varepsilon\big) \leqq \frac{1}{\varepsilon^2 n} \sum_{ab \leqq n} \frac{1}{F^2(b)} \sum_{\substack{b_2|b \\ b_2\in B}} f(b_2) \sum_{\substack{b_1|b \\ b_1\in B(b_2)}} f(b_1)$$

$$+\frac{1}{\varepsilon^2 n} \sum_{ab \leqq n} \frac{f^2(b)}{F^2(b)} \sum_{\substack{b_2|b \\ b_2\in B}} \frac{1}{f(b_2)} \sum_{\substack{b_1|b \\ b_1\in B(b_2)}} \frac{1}{f(b_1)} =: \Sigma_1 + \Sigma_2.$$

Observe that the quantities considered are determined by the common value of $f(p)=\varkappa>0$ on primes and the second sum Σ_2 can be obtained from Σ_1 by the substitution $f \to 1/f$. Hence an estimate of Σ_1 in terms of $\varkappa$ will be also valid for Σ_2 after the change $\varkappa \to 1/\varkappa$. It remains to deal with one of these sums. We have

$$\Sigma_1 \leqq \frac{1}{\varepsilon^2 n} \sum_{b_2\in B} f(b_2) \sum_{b_1\in B(b_2)} \frac{f(b_1)}{F^2([b_1,b_2])} \sum_{m \leqq n/[b_1,b_2]} \frac{F^2([b_1,b_2])}{F^2(m(\delta)[b_1,b_2])}.$$

According to Lemma 1 in virtue of $b_1b_2 \leqq n^{1-\delta_1/2}$ the inner sum can be majorized by

$$\frac{n}{[b_1,b_2]\delta_1 \log n} \exp\left\{ \sum_{p\leqq n^\delta} \frac{1}{p} + \frac{1}{(1+\varkappa)^2} \sum_{n^\delta < p \leqq n/[b_1,b_2]} \frac{1}{p} + \sum_{p|[b_1,b_2]} \frac{1}{p} \right\}$$
$$\ll \frac{ne(b_1)e(b_2)\delta^{1-1/(1+\varkappa)^2}}{[b_1,b_2]\delta_1}.$$

Hence using the estimate

$$e(b) \leqq \exp\left\{\frac{\omega(b)}{n^\delta}\right\} \leqq \exp\left\{\frac{1}{\delta n^\delta}\right\} \ll 1$$

for $n \geqq n_0(\delta)$, we obtain

$$\Sigma_1 \ll \frac{\delta^{1-1/(1+\varkappa)^2}}{\varepsilon^2\delta_1} \sum_{b_2\in B} f(b_2) \sum_{b_1\in B(b_2)} \frac{f(b_1)}{[b_1,b_2]F^2([b_1,b_2])}.$$

Further we denote $s=(b_1,b_2)$ and recall that all the prime factors of s are also greater than n^δ. Substituting sb_1 and sb_2 for b_1 and b_2, resp., we have

$$\Sigma_1 \ll \frac{\delta^{1-1/(1+\varkappa)^2}}{\varepsilon^2\delta_1} \sum_{s\leqq n} \frac{f^2(s)}{sF^2(s)} \sum_{b_2\leqq n} \frac{f(b_2)}{b_2F^2(b_2)} \sum_{b_1\in B(b_2)} \frac{f(b_1)}{b_1F^2(b_1)}. \tag{14}$$

Applying (1) for the subintervals $[N,2N]$ with $N=b_2n^{-\delta_3}$ we see that the inner sum of the last inequality can be estimated by

$$\delta_3 \log n \max_{n^{\delta_1/3}\leqq N\leqq n} \frac{1}{N} \sum_{b_1\leqq 2N} \frac{f(b_1)}{F^2(b_1)} \ll \frac{\delta^{1-\varkappa/(1+\varkappa)^2}}{\delta_1^2}.$$

The estimate (2) of Lemma 1 yields also

$$\sum_{s\leqq n} \frac{f^2(s)}{sF^2(s)} \sum_{b_2\leqq n} \frac{f(b_2)}{b_2F^2(b_2)} \ll \delta^{-\varkappa/(1+\varkappa)}.$$

Thus, the last two estimates and (14) imply

$$\Sigma_1 \ll \frac{\delta}{\varepsilon^2\delta_1^3} \to 0$$

as $\delta \to 0$.

Since the same estimate holds for Σ_2, the assertion (II) is established. The Theorem is proved.

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