

Review of Chapters 1-4 from Groups, Representations and Physics by H. F. Jones

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Recap

- We started this book at the beginning of this year (2026 01 30 to be exact).
- We had 17 seminars, with 11 of them covering the material from the chapters and 6 devoted to the exercises.
- We covered 4 chapters that comprise 77 pages of the book.
- We did 34 exercises.
- All the information about the seminars is in the seminar webpage, in either English (<https://web.vu.lt/ff/a.gelzinis/group-theory-in-physics-seminars/>) or Lithuanian (<https://web.vu.lt/ff/a.gelzinis/grupiu-teorijos-fizikoje-seminarai/>).

Introduction

- Symmetry is important in physics, group theory describes symmetries, so it should be useful...

Chapter 1 – definition of a group

- A group G is a set of elements $\{a, b, \dots\}$ and a law of composition (or multiplication), which assigns to each ordered pair $a, b \in G$ an element $ab \in G$ (this is closure – G0). The composition law needs to satisfy the following requirements:

- G1 (associative law).

$$a(bc) = (ab)c.$$

- G2 (unit element). There exists such an element e that

$$ae = ea = a.$$

- G3 (existence of inverse). For all group element there exists an inverse element:

$$aa^{-1} = a^{-1}a = e$$

- For Abelian groups $ab = ba$.

Chapter 1 – Simple point groups

- The Cyclic Group C_n , which is the symmetry group of rotations of a regular polygon with n directed sides.

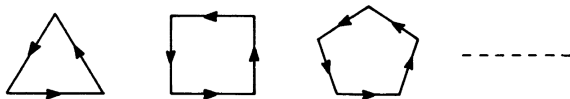


Figure 1.1 Regular polygons with directed sides which are transformed into themselves by the rotations of the group C_n .

- The generating element is denoted by c and all rotations can be constructed by applying c in succession several times. This is expressed as $C_n = \text{gp} \{c\}$.
- C_n includes n elements (it is of the order n).

Chapter 1 – Simple point groups

- The Dihedral Group D_n , which is the symmetry group of rotations of a regular polygon with n undirected sides.

Group D_3

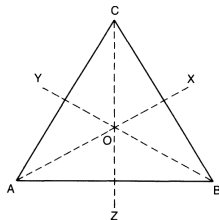


Figure 1.2 Equilateral triangle with undirected sides which is transformed into itself by the rotations of the group D_3 .

- This group includes elements of C_n and also various “flips” (like around CO axis). Denoting one such flip as b , we can write $D_n = \text{gp} \{c, b\}$.
- Generally, $D_n = \text{gp} \{c, b\}$ with $c^n = b^2 = (bc)^2 = e$. This group is of the order $2n$, canonical form is $\{e, c, c^2, \dots, b, bc, bc^2, \dots\}$.

Chapter 1 – Permutation Group S_n

- This group consists of permutations of n objects and it is of the order $n!$
- It is conveniently described in cycle notation. For example, permutation $1 \rightarrow 2 \rightarrow 4$ and $3 \rightarrow 3$ can be written as

$$(1 \ 2 \ 4) (3)$$

and (3) can be skipped.

- Successive permutations can be simplified:

$$(2 \ 3) (1 \ 3 \ 2) = (1 \ 2)$$

Chapter 1 – Cayley's theorem

- It states that every finite group of order n can be considered as (is isomorphic to) a subgroup of S_n .
- This arises from the fact that for a finite group G , multiplication of all the elements $\{a_i\}$ by a given element g permutes them:

$$\{ga_1, \dots, ga_n\} = \{a_{\pi_1}, \dots, a_{\pi_n}\}.$$

Thus to each element g there is a corresponding permutation $\Pi(g)$ of S_n :

$$g \rightarrow \Pi(g) = \begin{pmatrix} 1 & 2 & \dots & n \\ \pi_1 & \pi_2 & \dots & \pi_n \end{pmatrix}.$$

Chapter 1 – Alternating group A_n

- It consists of even permutations of S_n .
- A single transposition, or a 2-cycle (ij) , is odd. Any r -cycle can be written as a product of $r - 1$ overlapping 2-cycles. Therefore, a cycle of length r is even/odd according to if r is odd or even.
- The order of A_n is half that of S_n .

Chapter 2 – equivalence relation

- It is a set of rules that determine if two elements a, b of some set S are equivalent, written as $a \sim b$. Three properties must be satisfied:
 - reflexivity, $a \sim a$.
 - symmetry, $a \sim b \implies b \sim a$.
 - transitivity, $a \sim b$ and $b \sim c \implies a \sim c$.

Chapter 2 – conjugacy

- Two elements $a, b \in G$ are conjugate if there exists element $g \in G$ such that $a = gbg^{-1}$. The element g is called the conjugating element.
- Conjugacy is an equivalence relationship:
 - $a \sim a$, since $a = eae^{-1}$;
 - $a \sim b \Rightarrow a = gbg^{-1} \Rightarrow b = g^{-1}ag$;
 - $a \sim b \Rightarrow a = gbg^{-1}$ and $b \sim c \Rightarrow b = hch^{-1}$, thus
 $a = g(hch^{-1})g^{-1} = (gh)c(gh)^{-1} \Rightarrow a \sim c$.

Chapter 2 – equivalence and conjugacy classes

- With any equivalence relation we can partition a set into disjoint equivalence classes. The equivalence class of an element a , denoted as (a) , is the set of elements equivalent to a :

$$(a) = \{b | b \sim a\} .$$

- Conjugacy class of an element is written as

$$(a) = \{b | b = gag^{-1}, g \in G\} .$$

Chapter 2 – subgroups

- A subgroup H of a group G is such a subset of G that forms a group under the composition law of G .
- For finite groups, this is equivalent to saying that H should be closed under the composition law of G .
- The identity element e forms a subgroup, as does the whole group G . A proper subgroup, denoted by $H < G$, is a subgroup that is not one of the two aforementioned cases.

Chapter 2 – cosets

- For $H < G$, the left coset of an element $g \in G$, denoted as gH , is defined as the set of elements obtained by multiplying all the elements of H on the left by g :

$$gH = \{gh_1, \dots, gh_r\}.$$

- Coset is not a group.
- The cosets either completely overlap or are completely disjoint.
- The set of cosets is denoted as G/H .

Chapter 2 – Lagrange's theorem

- The number of elements of group G , $[g]$, can be expressed as

$$[g] = s [h] ,$$

with $[h]$ being the order of any subgroup of G and s being an integer.

- In other words, all elements of G can be grouped into s cosets with $[h]$ elements.
- Groups of prime order have no proper subgroups.

Chapter 2 – Normal subgroups

- A normal, or invariant, or, more clearly, self-conjugate subgroup H of G satisfies

$$gHg^{-1} = H, \quad \forall g \in G.$$

This means that

$$h \in H \quad \Rightarrow \quad ghg^{-1} \in H, \quad \forall g \in G$$

and not

$$ghg^{-1} = h, \quad \forall g \in G.$$

- An equivalent formulation is that $gH = Hg$.

Chapter 2 – quotient group

- The set of cosets G/H can be endowed with a group structure by a suitable definition of the product of two cosets.
- Let us define the product of two cosets g_1H and g_2H as a coset g_1g_2H :

$$(g_1H) (g_2H) = g_1g_2H.$$

- G0: $g_1g_2H \in G/H$ since $g_1g_2 \in G$.
- G1: $(g_1H) [(g_2H) (g_3H)] = [(g_1H) (g_2H)] (g_3H)$ due to associativity of G .
- G2: coset $E = eH = H$ acts as the identity, since $(eH) (gH) = (eg)H = gH$.
- G3: Inverse of gH is $g^{-1}H$ since $(gH) (g^{-1}H) = (gg^{-1})H = eH = E$.

Chapter 2 – direct product of groups

- A group G is the direct product of its subgroups A and B , denoted as $G = A \times B$, if
 - all elements of A commute with all elements of B .
 - every element of G can be written in a unique way as $g = ab$.
- Product of $g_1 = a_1b_1$ and $g_2 = a_2b_2$ is

$$g_1g_2 = (a_1b_1)(a_2b_2) = (a_1a_2)(b_1b_2).$$

- A and B are normal subgroups of G .

Chapter 2 – homomorphism

- Definition from Viktor's slides: homomorphism is a function with domain (what is being mapped) and co-domain (on what is being mapped) being the same type of algebraic structures (group, field, ring, vector space, module). Homomorphism preserves operations defined in these algebraic structures.
- Definition from the book: A homomorphism is a mapping f from one set A to another set B (written $f : A \rightarrow B$) preserving some structure or other.

Chapter 2 – homomorphism

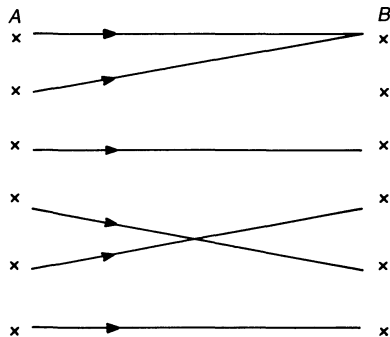


Figure 2.2 General mapping f from one set A to another set B .

- Image of f , denoted as $\text{Im } f$ or $f(A)$, consists of those elements of B which are maps of elements of A :

$$\text{Im } f = \{b \in B \mid b = f(a)\}.$$

Chapter 2 – homomorphism

- Mapping can be *onto* or *surjective* (onto entire of something), as opposed to *into* or *injective* (into some part of something).
- $f : A \rightarrow B$ is a group homomorphism if, for all $a_1, a_2 \in A$,

$$f(a_1 a_2) = f(a_1) f(a_2) .$$

- In case of group homomorphism the image of mapping forms a subgroup of B .
- Kernel of the homomorphism, denoted as $\text{Ker } f$ or $f^{-1}(e_B)$, is a normal subgroup consisting those elements of A which are mapped onto the identity element in B :

$$\text{Ker } f = \{a \in A | f(a) = e_B \in B\} .$$

Chapter 2 – isomorphism theorem

- If $f : G \rightarrow G'$ is a homomorphism of G into G' , with kernel K , then

$$\text{Im } f \cong G/K.$$

Here $\cong$ means isomorphic and G/K is the quotient group (a set of cosets).

- Quoting Viktor's slides again, isomorphism is bijective homomorphism.
- Bijective means 1-to-1 and onto.
- 1-to-1 means $\forall x, x' \in X, f(x) = f(x') \Rightarrow x = x'$.
- Onto means $\forall y \in Y, \exists x \in X, y = f(x)$.
- Endomorphism is homomorphism where a domain coincides with a co-domain.
- Automorphism is bijective endomorphism.

Chapter 3 – C_3 group representation

- The group $C_3 = \text{gp} \{c\}$, $c^3 = e$ can be represented by matrices:

$$D(e) = R(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad D(c) = R\left(\frac{2\pi}{3}\right) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$$D(c^2) = R\left(\frac{4\pi}{3}\right) = R\left(\frac{2\pi}{3}\right) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Here

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a rotation matrix.

Chapter 3 – definition of representation

- A representation of dimension n of the abstract group G is defined as a homomorphism $D : G \rightarrow \text{GL}(n, \mathbb{C})$, the group of non-singular $n \times n$ matrices with complex entities.
- Such mapping $g \mapsto D(g)$ must preserve the group structure:

$$D(g_1 g_2) = D(g_1) D(g_2) .$$

- For such mapping the matrices must be invertible, $D(g^{-1}) = (D(g))^{-1}$, thus the matrices have to be non-singular.
- The matrices for $D(c)$ and $D(c^2)$ for C_3 are real and orthogonal. Their determinant is equal to 1 (proper rotations). The group of such matrices is $\text{SO}(3)$.
- Representation is “faithful” if it is isomorphism. It is stated that this is the case of the kernel of the homomorphism consists only of the identity element: $\text{Ker } D = e \in G$. Example of an “unfaithful” representation is the trivial representation,

Chapter 3 – transformation of QM wavefunction

- As an example we will consider a p-state ($l = 1$) electron.

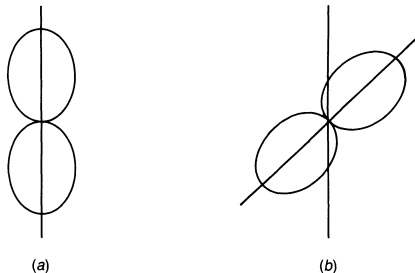


Figure 3.3 Polar plot of the wavefunction of a p-state electron (a) before and (b) after a rotation of the whole system.

Chapter 3 – transformation of QM wavefunction

- Lets assume the system is rotated. Before the rotation we had the wavefunction $\psi(\vec{x})$, and after the rotation we have $\psi'(\vec{x})$.
- The value of the new wavefunction at a rotated point is the same as that of the old wavefunction at the original point, thus

$$\psi'(\vec{x}') = \psi(\vec{x}).$$

Here $\vec{x}' = \hat{R}\vec{x}$. Therefore,

$$\psi'(\vec{x}') = \psi(\hat{R}^{-1}\vec{x}').$$

- We can relabel $\vec{x}' \rightarrow \vec{x}$, thus

$$\psi'(\vec{x}) = \psi(\hat{R}^{-1}\vec{x}).$$

- Rotation in the physical space has induced a transformation in the space of wavefunctions.

Chapter 3 – transformation of QM wavefunction

- For hydrogen atom, the energy eigenfunctions are labeled, by quantum numbers n (for $E_n = E_1/n^2$), l (for $\vec{L}^2 = l(l+1)\hbar$) and m (for $L_z = m\hbar$).
- After rotation, n and l quantum numbers do not change, only m changes.
- Thus, we can express the transformed wavefunction as a linear superposition of the told wavefunctions:

$$u'_{nlm}(\vec{x}) = \sum_{m'} D_{m'm}^l(\hat{R}) u_{nlm'}(\vec{x}) .$$

- The matrix $D_{m'm}^l(\hat{R})$ is a square matrix of dimension $(2l+1) \times (2l+1)$.
- These matrices satisfy

$$D(\hat{R}_1 \hat{R}_2) = D(\hat{R}_1) D(\hat{R}_2) ,$$

thus they constitute a representation, of dimension $2l+1$, of the group of 3-dimensional rotations.

Chapter 3 – transformation of QM wavefunction

The book states that when considering some symmetry transformation acting on some QM system we need to know how to:

- 1 Classify and enumerate the possible representations.
- 2 Combine representations.
- 3 Relate the representations of a subgroup to those of the whole group.

Chapter 3 – equivalence of representations and character

- Two $[n]$ representations $D^{(1)}$ and $D^{(2)}$ of a group G are equivalent if all the matrices $D^{(1)}(g)$ and $D^{(2)}(g)$ are related by the same similarity transformation:

$$D^{(1)}(g) = S D^{(2)}(g) S^{-1}.$$

- The character of a group representation D of a group G is the set $\chi = \{\chi(g) | g \in G\}$, where $\chi(g)$ is the trace of the representation matrix $D(g)$:

$$\chi(g) = \text{Tr}(D(g)).$$

- Traces of matrices related by a similarity transformation are equal, thus characters of equivalent representations are the same.

Chapter 3 – reducible representations

- A representation of dimension $n + m$ is said to be reducible if $D(g)$ is of the form

$$D(g) = \begin{pmatrix} A(g) & C(g) \\ 0 & B(g) \end{pmatrix}.$$

Here submatrices A , C and B are of dimensions $m \times m$, $m \times n$ and $n \times n$, respectively.

- Multiplying two such matrices we get

$$D(g) D(g') = \begin{pmatrix} A(g) A(g') & A(g) C(g') + C(g) B(g') \\ 0 & B(g) B(g') \end{pmatrix}.$$

Thus $\{A(g)\}$ and $\{B(g)\}$ constitute $[m]$ and $[n]$ representations of G .

Chapter 3 – reducible and irreducible representations

- For finite groups C can be taken as a null matrix, then the representation is completely reducible or decomposable:

$$D(g) = A(g) \oplus B(g) .$$

- Such decomposition process can be continued until we reach the level of irreducible representations.
- The irreducible representations can be classified (by characters) and enumerated.

Chapter 3 – vector space

A vector space V over the field of complex numbers is a set of elements $\{v\}$ with two operations.

Addition satisfies:

- A0 (closure): $u + v \in V, \forall u, v \in V$.
- A1 (associativity): $u + (v + w) = (u + v) + w$.
- A2 (existence of null vector): in V there exists $\mathbf{0}$, such that $u + \mathbf{0} = u$.
- A3 (existence of inverse): for all $u \in V$, there exists $-u$, such that $u + (-u) = \mathbf{0}$.
- A4 (commutativity): $u + v = v + u$.

V forms an Abelian group under the addition as the composition.

Chapter 3 – vector space

For vector space, multiplication by complex numbers satisfies:

- B0: $a\mathbf{u} \in V, \forall a \in \mathbb{C}, \mathbf{u} \in V$.
- B1: $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
- B2: $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.
- B3: $a(b\mathbf{u}) = (ab)\mathbf{u}$.
- B4: $1 \cdot \mathbf{u} = \mathbf{u}$.

Chapter 3 – more definitions

- A set of vectors $\{\mathbf{e}_i\}$, $i = 1, \dots, m$, is linearly independent if there is no non-trivial combination which gives the null vector:

$$\sum_{i=1}^m \lambda_i \mathbf{e}_i \quad \Rightarrow \quad \lambda_i = 0.$$

- A linearly independent set of vectors $\{\mathbf{e}_i\}$, $i = 1, \dots, m$ forms a basis of a vector space V if they span the space, thus all $\mathbf{u} \in V$ can be written as

$$\mathbf{u} = \sum_{i=1}^m u_i \mathbf{e}_i.$$

- A vector space V is n -dimensional if it has as basis of n vectors.

Chapter 3 – linear transformations

- A map $T : V \rightarrow V$ is linear if

$$T(\alpha \mathbf{u} + \beta \mathbf{v}) = \alpha (T\mathbf{u}) + \beta (T\mathbf{v}).$$

- For a chosen basis $\{\mathbf{e}_i\}$ of V , the map T has a realization as a matrix D_{ij} , which can be defined from the transformation of the basis vectors:

$$T\mathbf{e}_j = D_{ij}\mathbf{e}_i.$$

- From $T\mathbf{u} = \mathbf{v}$, vector elements transform as

$$v_i = D_{ij}u_j.$$

- Upon change of basis

$$D' = SDS^{-1}.$$

Chapter 3 – G -module and submodule

- Every element $g \in G$ can be mapped to a linear transformation $T(g)$, which satisfies

$$T(g'g) = T(g')T(g).$$

A vector space in which a group G acts in this way is called G -module.

- The reducibility corresponds to an existence of a submodule (subspace U of V , which is closed under the action of the group):

$$u \in U \subset V \quad \Rightarrow \quad T(g)u \in U.$$

The basis $\{e_i\}$ with $i = 1, \dots, m$ can be extended by $\{e_i\}$ with $i = m + 1, \dots, m + n$ to form the basis of V .

Chapter 3 – scalar product

- Scalar product is a map $V \times V \rightarrow \mathbb{C}$, which assigns to each ordered pair $\mathbf{u}, \mathbf{v} \in V$ a complex number $(\mathbf{u}, \mathbf{v})$ with the following properties:
 - S1 (hermicity): $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})^*$;
 - S2 (linearity): $(\mathbf{w}, \alpha \mathbf{u} + \beta \mathbf{v}) = \alpha (\mathbf{w}, \mathbf{u}) + \beta (\mathbf{w}, \mathbf{v})$;
 - S3 (positivity): $(\mathbf{u}, \mathbf{u}) \geq 0$ with equality only if $\mathbf{u} = \mathbf{0}$.
- Norm of the vector is $\|\mathbf{u}\| = \sqrt{(\mathbf{u}, \mathbf{u})}$.
- Orthonormal basis satisfies $(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}$.

Chapter 3 – unitary transformations

- A linear transformation is unitary if

$$(Tu, Tv) = (u, v), \quad \forall u, v \in V.$$

When T has an inverse, this means

$$(Tu, v) = (u, T^{-1}v).$$

- In matrix form

$$D_{ij}^* = (D^{-1})_{ji}.$$

- W is an orthogonal complement of a subspace U if

$$W = \{w \in V \mid (w, u) = 0, \quad \forall u \in U\}.$$

Complete reducibility then corresponds to the closure of W as well as U under the action of the group.

Chapter 3 – Maschke's theorem

- It states that all reducible representations of a finite group are completely reducible.
- Group invariant scalar product is defined by

$$\{v, v'\} = \frac{1}{[g]} \sum_{g \in G} (T(g)v, T(g)v') .$$

Chapter 4 – Schur's lemmas

- The first Schur's lemma states that any matrix which commutes with all the matrices of an irreducible representation must be a multiple of the unit matrix:

$$BD(g) = D(g)B, \quad \forall g \in G \quad \Rightarrow \quad B = \lambda 1.$$

- The second Schur's lemma states that for two irreducible representations D and D' we have

$$BD(g) = D'(g)B, \quad \forall g \in G \quad \Rightarrow \quad B = \hat{0}.$$

- The fundamental orthogonality theorem states that

$$\sum_g D_{ir}^{(\mu)}(g) D_{sj}^{(\nu)}(g^{-1}) = \frac{[g]}{n_\mu} \delta^{\mu\nu} \delta_{ij} \delta_{rs}.$$

Here n_μ is the dimensionality of $D^{(\mu)}$. It follows that $\sum_\mu n_\mu^2 \leq [g]$.

Chapter 4 – orthogonality of characters

- Orthogonality relation for characters:

$$\sum_g \chi^{(\mu)}(g) \underbrace{\chi^{(\nu)}(g^{-1})}_{\chi^{(\nu)*}} = [g] \delta^{\mu\nu}.$$

- Scalar product of characters:

$$\langle \varphi, \chi \rangle = \frac{1}{[g]} \sum_g \varphi(g) \chi(g^{-1}) = \langle \chi, \varphi \rangle.$$

- After some steps it follows that the number of inequivalent irreducible representations is less than or equal to the number of conjugacy classes: $r \leq k$. This turns out to be equality.

Chapter 4 – decomposition of reducible representation

- For finite group any reducible representation is completely reducible into a direct sum of irreps:

$$D = \sum_{\oplus} a_{\nu} D^{(\nu)}.$$

The coefficients a_{μ} can be obtained from the orthogonality of characters:

$$a_{\mu} = \frac{1}{[g]} \sum_g \chi(g) \chi^{(\mu)}(g^{-1}).$$

Chapter 4 – regular representation

- The $[g] \times [g]$ permutation matrices $D_{jr}(g)$ form a $[g]$ -dimensional representation of the group, called the regular representation.
- For example, in C_3 :

$$D(c) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- Decomposing the regular representation into its irreducible components:

$$D = \sum_{\oplus} a_{\nu} D^{(\nu)}, \quad a_{\mu} = n_{\mu}.$$

Chapter 4 – construction of the character table

- When constructing the character table we utilize:
 - The number of irreps = number of conjugacy classes, $r = k$.
 - $\sum_{\mu} n_{\mu}^2 = [g]$.
 - orthogonality: $\sum_i k_i \chi_i^{(\mu)} \chi_i^{(\nu)*} = [g]$.
 - any other information (like that for 1D reps the characters must mimic the group multiplication properties).

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 - any other information (like that for 1D reps the characters must mimic the group multiplication properties).

Chapter 4 – representations of the direct products of the groups

- Lets assume that we have two particles with wavefunctions $\psi_a(\vec{r})$ and $\varphi_b(\vec{r})$. The composite system would be $\psi_{ab}(\vec{r}_1, \vec{r}_2) = \psi_a(\vec{r}_1) \varphi_b(\vec{r}_2)$.
- The composite system would transform as
$$\psi'_{ab}(\vec{r}_1, \vec{r}_2) = D_{ac}^{(\mu)}(g) D_{bd}^{(\nu)}(g) \psi_{cd}(\vec{r}_1, \vec{r}_2).$$
- Representation of the direct product of the group would then be $D_{ab;cd}^{(\mu \times \nu)}(g) = D_{ac}^{(\mu)}(g) D_{bd}^{(\nu)}$. In matrix notation we would have $D^{(\mu)} \otimes D^{(\nu)}$.
- Such product can be decomposed as $D^{(\mu)} \otimes D^{(\nu)} = \sum_{\oplus} a_{\sigma} D^{(\sigma)}$. This is called the Clebsch-Gordan series.